LINEAR MULTIVARIABLE CONTROL SYSTEMS

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To

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PREFACE

Since about 1966, the modern multivariable control theory was developed along different parallel lines. Before this time the time domain and frequency domain techniques of classical control theory for single input single output (SISO) systems were well established and practically tested. Naturally the first trend in the development of multivariable control system (mvcs) theory was to extend the classical theory to the multi-input multioutput case. The state space description introduced in the later stage of classical control theory was well suited to accommodate the mvcs description also. The transfer functions and polynomials of frequency domain, however, were to be replaced by transfer matrices and polynomial matrices. The matrix algebra and geometrical space theory then became the mathematical basis of the developments in mvcs state space techniques and the polynomial matrix theory became the basis of differential operator and transfer matrix techniques. The graphical techniques in frequency domain continued to appear in the mvcs as direct extension of classical control theory by making a mvcs represent a noninteractive combination of SISO systems.

Some of the developments in mvcs theory were sufficiently well established to make appearance of a few textbooks by 1972-74. However there was a vast difference in their approaches and notations in spite of the common stream of basic principles and objectives. A mixed approach taken by

Wolovich[W-3] using the matrix algebra for state space techniques and polynomial matrix theory for differential operator and transfer matrix techniques was found appealing and promising. The same is accepted in this book. However the field itself being a developing one still, there is always a lot of scope for modifications and additions justifying a new textbook.

This book is intended to give a basic course on mvcs for the advanced under graduate or the graduate students. The aim is to present an easily understandable logical stream of development to enable the reader to solve the basic problems of analysis and synthesis of continuous, linear, time invariant deterministic mvcs.

The prerequisit for understanding this book is the knowledge of classical control theory and fundamentals of the theory of at least the scalar matrices. Mathematical techniques in polynomial matrices ar. included in the book as and when needed. For any more details of the used mathematical techniques the reader is referred to Gantmatcher[G-1] and Cullen[C-1].

The book reports certain developments in mvcs theory with modest addition of original contribution. Hence I feel it very important to, first, express my gratitude to all those investigators whose efforts have resulted in these developments. I also thank the reviewers, to whom the material of this book

was communicated, for their valuable comments. I acknowledge with thanks the financial aid given by the Curriculum Development Unit of the Indian Institute of Technology, Bombay. Thanks are also due to Mr. S. A. Talawdekar for converting my untidy manuscript into a neat typescript, without delay.

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CHAPTER - 1

INTRODUCTION

The multivariable control system has p outputs and m inputs where p and/or m are larger than one. Each of the outputs may, in general, be affected by any input. This is called interaction. This is the property which makes theory of multivariable control system different from the classical and modern state space theory of single input single output systems (SISO).

There are many practical systems which are controlled by more than one inputs and in which more than one outputs are of interest. Many chemical processes are of this nature which need simultaneous control of pressure, temperature, concentration etc. by commanding a number of inputs. Aircrafts and spacecrafts are other typical examples where the movement is controlled by three inputs. Power generators, atomic reactors, jet engines are some more examples where a number of variables are to be controlled from a number of inputs causing interaction. Design and analysis of such systems cannot be done directly using the SISO techniques, because of the interaction.

As is well known the classical control theory for SISO systems is characterised by graphical methods like Bode diagrams root loci., Nyquist plots etc. giving more insight with less amount of calculations. With the availability of fast

computing facilities, the state space analysis using compact matrix representation and revealing detailed system behaviour became popular. The state space form of representation is suitable for multivariable control systems also and is desirable for detailed analysis of stability, controllability and observability. But, whenever in the practical system the complete system state is not measurable, such a detailed representation is not useful for design of compensation. In that case, frequency domain design techniques, using transfer matrix representation and dealing with only the input output characteristics, are more convenient than time domain design techniques. However according to the modern trend of prefering exact computational methods to the approximate graphical methods, this book will deal with only the computational methods even in the frequency domain.

In general, time domain techniques are more suitable for complete analysis and frequency domain techniques are more suitable for practical compensation problems. Hence it is necessary to learn all the techniques of representations and methods of conversion from any one to another and vice versa.

The book can be devided in four parts. The first one, deals with representation and solution. The second part describes the important system properties like controllability observability and reproducibility. The third part presents methods of transformation among the different forms of representation and fourth one gives compensation techniques. While the material on the first three parts is well established, the

design part which is still under a stage of development includes a number of original modifications to the existing techniques. The number of chapters in first three parts are 3, 3 and 2 respectively. The ninth chapter gives time domain design technique which is convenient for linear state variable feedback (l.s.v.f.) compensation. Arbitrary pole allocation for achieving required degree of stability can be most directly done by l.s.v.f. In chapter 10 it is shown that dynamic feedback compensation in frequency domain can in general provide arbitrary pole placement. Under certain constraints it can directly be converted into l.s.v.f. It can also give non-interactive control. Model matching compensation, which is more difficult to achieve, but is more desirable, is described in the 11th chapter. Other existing techniques of design and analysis are compared to those discussed in this book in chapter 12. It also gives references to the literature pertaining to these techniques for interested leader. The later part of this chapter gives concluding remarks about the coverage of multivariable control theory in this book and comments on contributions.

The mathematical techniques used are introduced as and when they are needed in the text and are generally illustrated by numerical examples. A number of numerical examples are solved to illustrate the use of analysis and design algorithms. Exercise problems are also given. References to literature are made by capital letter followed by a number in brackets above the lines in the text.

CHAPTER - 2

REPRESENTATION

2.1 Introduction

There are three major methods of representation of multi-variable control systems (MVCS).

i) Differential Operator representation [D.O.].

ii) State Space representation [S.S.].

iii) Transfer Matrix representation [T.M.].

The first two are time domain representations and the third one is frequency domain representation.

In classical control theory frequency domain representation is done using transfer function. The graphical techniques of analysis and synthesis make use of root loci, Nyquist plots Bode plots etc. As is well known in the classical control theory the advantages of frequency domain techniques over time domain techniques are the following :

i) Ease of getting transfer function description by experimental frequency response of the system components.

ii) Graphical methods used for analysis and design.

iii) Less amount of computation.

iv) Good insight into the problem due to graphical techniques.

Increased use of digital computer has eliminated the point of less amount of calculations from the list of desired

qualities. Hence time domain techniques are more popular in modern control theory. It is the time domain response that we are practically interested in. So if it can be obtained directly without going through frequency domain transformations, it is more advisable. As stated above the time domain representation can be in two forms : (i) D.O. and (ii) S.S. In (i) the system description is in a more compact form than (ii). But S.S. is a more systematic representation dealing with constant matrices and useful for use of either analog or digital computer. Moreover it gives more explicit description of the system.

The third method of representation (T.M.) is also used in MVCS for those advantages of frequency domain representation quoted above which are still valid.

Particular techniques of analysis and synthesis of MVCS are suited to particular forms of representations. Hence transfer between different forms of representations becomes an important part of the theory of analysis of MVCS.

2.2 Differential Operator Representation

A linear, time invariant and continuous dynamical multivariable control system can be represented by the following set of ordinary linear differential equations :

$$P(D)\mathbf{z}(t) = Q(D)\,\mathbf{u}(t)$$

and $$\mathbf{y}(t) = R(D)\mathbf{z}(t) + W(D)\mathbf{u}(t) \qquad \ldots(2.2.1)$$

where P, Q, R and W are matrix differential operators of dimensions q x q, q x m, p x q and p x m respectively. $\mathbf{z}(t)$,

$u(t)$, $y(t)$ are vectors with time function elements. They represent the system variables, the inputs and the outputs respectively. As shown in chapter 8, the system order n is,

$$n = \deg |p(D)| \qquad \ldots(2.2.2)$$

Single input single output system is a particular case of the above D.O. description in which $u(t)$ and $y(t)$ are scalars, not vectors. Hence SISO systems can better be referred to as scalar systems to differentiate them from the multivariable control systems.

Most of the practical systems can be adequately described by the linear differential equation (2.2.1), at least after removing the nonlinearities by purturbation techniques.

The D.O. representation can be directly converted to T.M. representation by Laplace transformation assuming zero initial conditions.

2.3 <u>State Space Representation</u> :

This is a more detailed description of the system in time domain. This is obtained by converting the differential equations of the D.O. form into n first order differential equations. They can be represented in the compact form by the following matrix equations.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Eu \end{aligned} \qquad \ldots(2.3.1)$$

Here, x is an n dimensional vector of state variables, called the state vector. u is the m dimensional input vector and y is the p dimensional output vector. A, B, C, E are n x n, n x m, p x n and p x m scalar matrices called evolution matrix, control matrix, observation matrix and direct transmission matrix respectively. The state space representation can be specified by the quadruple (A, B, C, E).

Rosenbrock[R-2] combines the two equations of (2.3.1) and after Laplace transformation represents the system by the matrix equation

$$\begin{bmatrix} sI_n - A & B \\ -C & E \end{bmatrix} \begin{bmatrix} x \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix} \qquad \ldots(2.3.2)$$

and calls the (n + p) x (n + m) matrix

$$\begin{bmatrix} sI_n - A & B \\ -C & E \end{bmatrix} \qquad \ldots(2.3.3)$$

as the system matrix. This type of state space description is also common in literature.

The differential operator quadruple (P, Q, R, W) corresponding to the S.S. representation is obviously (DI-A, B, C, E). Rosenbrock's system matrix corresponding to the D.O. representation is then obviously given by

$$\begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix} \qquad \ldots(2.3.4)$$

The state space representation reveals the dynamics of the system more explicitly and hence is necessary for studying controllability and observability.

2.4 Transfer Matrix Representation

In the scalar case the system is represented by a transfer function in the frequency domain. This is the ratio of Laplace transform of the output to the Laplace transform of the input, this ratio being characteristic of the given system. In a MVCS there are many outputs and many inputs. Hence it can be represented by pm transfer functions between the p outputs and m inputs, calculated by considering one input at a time. The transfer matrix relation between the Laplace transformed output and input vectors can be written as

$$y(s) = T(s)\, u(s) \qquad \ldots(2.4.1)$$

where the pm elements of $T(s)$ are transfer functions, i.e. ratios of polynomials in s. $T(s)$ can always be represented as $R(s)\ P^{-1}(s)$ where $R(s)$ and $P(s)$ are matrices with their elements in the form of polynomials. $T(s)$ can also be represented as $R(s)/p(s)$ where $R(s)$ is a polynomial matrix and $p(s)$ is a polynomial.

The polynomial matrices differ from the scalar matrices in an important aspect that, the elements of a scalar matrix belong to a field, whereas the elements of a polynomial matrix belong to a commutative ring with identity. The later has all the properties of a field except that the inverse of a polyno- mial of degree one or more is not a polynomial.(See)Appendix).

Any p x m polynomial matrix R(s) can be written as

$$R(s) = R_0 + s\,R_1 + s^2\,R_2 + \dots s^n\,R_n \qquad \dots(2.4.2)$$

where R_i are p x m scalar matrices and n is the degree of highest degree element of R(s).

The transfer matrix corresponding to S.S. representation (2.3.1) is,

$$T(s) = C\,(sI-A)^{-1}\,B + E \qquad \dots(2.4.3)$$

2.5 <u>Equivalent Transformations</u>

The state space representation involves scalar matrices and the other two representations involve polynomial matrices. More than one equivalent representations in each of these categories can be obtained through equivalent transformations of these matrices. Such equivalent transformations are useful to bring these matrices in some specially structured forms called canonical forms, suitable for analysis and design. In general two m x n matrices M and N are equivalent iff,

$$P\,M\,Q = N \qquad \dots(2.5.1)$$

where P and Q are nonsingular square matrices of orders m and n respectively. Obviously $\rho(M) = \rho(N)$ ...(2.5.2)
ρ, denoting the rank.

In the state space representation we can define a state vector $\bar{x}(t)$ equivalent to the state vector x(t) by the following relation;

$$\bar{x}(t) = Qx(t) \qquad \ldots(2.5.3)$$

where Q is a nonsingular matrix.

Under this transformation the state Eqns. (2.3.1) become

$$Q^{-1}\dot{\bar{x}} = AQ^{-1}\bar{x} + Bu$$

and
$$y = CQ^{-1}\bar{x} + Eu$$

Thus the transformed state equations can be written as

$$\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{B}u$$
$$y = \bar{C}\,\bar{x} + \bar{E}u \qquad \ldots(2.5.4)$$

with
$$\bar{A} = QAQ^{-1}$$
$$\bar{B} = QB$$
$$\bar{C} = CQ^{-1}$$
$$\bar{E} = E \qquad \ldots(2.5.5)$$

That is, the quadruple (A, B, C, E) gets transformed into an equivalent quadruple ($\bar{A}$ $\bar{B}$ $\bar{C}$ $\bar{E}$). Following equivalence properties are useful in analysis and design :

i) Two equivalent S.S. representations have the same characteristic polynomials as,

$$\begin{aligned} |sI-\bar{A}| &= |sI-QAQ^{-1}| \\ &= |Q[sI-A]Q^{-1}| \\ &= |Q|.|sI-A|.|Q^{-1}| \\ &= |sI-A| \end{aligned} \qquad \ldots(2.5.6)$$

Further it is easy to show, using expansion of e^{At} that,

ii) The state transition matrices of two equivalent representations are equivalent i.e.

$$e^{At} = Q\, e^{\bar{A}t}\, Q^{-1}$$

iii) The two equivalent representations have the same transfer matrix, i.e.

$$T(s) = C(sI-A)^{-1} B = \bar{C}\,(sI-\bar{A})^{-1}\,\bar{B} \qquad \ldots(2.5.7)$$

iv) Two equivalent state space representations have the same properties with respect to controllability and observability.

Techniques of equivalent transformations are useful in frequency domain also, to bring the polynomial matrices into some convenient canonical forms.

Two p x m polynomial matrices R(s) and $\bar{R}(s)$ are said to be equivalent iff either can be obtained from the other by elementary operations defined below :

1) Interchange any two rows (columns).
2) Multiply any row (column) by a nonzero scalar.
3) Add to any row (column) any other row (column) multiplied by a polynomial ...(2.5.8)

The relation between R(s) and $\bar{R}(s)$ can then be written as

$$\underset{p\times m}{R(s)} = \underset{p\times p}{U_L(s)}\ \underset{p\times m}{\bar{R}(s)}\ \underset{m\times m}{U_R(s)} \qquad \ldots(2.5.9)$$

Note that premultiplication by $U_L(s)$ corresponds to row operations and postmultiplication by $U_R(s)$ corresponds to column

operations. The square polynomial matrices $U_L(s)$ and $U_R(s)$ are called unimodular matrices. They can be obtained from identity matrix by elementary operations and their determinants are nonzero scalars. Thus obviously their inverses also are polynomial matrices,

$$\text{If } R(s) = U_L(s)\, R_1(s) \qquad \ldots(2.5.10)$$

$R(s)$ is called row equivalent to $R_1(s)$ and if

$$R(s) = R_2(s)\, U_R(s) \qquad \ldots(2.5.11)$$

$R(s)$ is called column equivalent to $R_2(s)$.

2.6 Canonical Forms

The useful canonical forms in the state space representation are,

1) Diagonal Form
2) Jordan Form
3) Companion Form.

1) It can be proved that any n x n matrix A with nonrepeating eigenvalues can be converted into its equivalent diagonal form

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \qquad \ldots(2.6.1)$$

where $\lambda_1 \ldots \lambda_n$ are the n eigenvalues of $\bar{A}$. Diagonal matrix is easy to invert. Moreover it is a convenient form to form noninteracting systems.

2) The Jordan form can be given as

$$\begin{bmatrix} \bar{A}_1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \bar{A}_m \end{bmatrix} \qquad \ldots(2.6.2)$$

with

$$\bar{A}_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \lambda_i \end{bmatrix} \qquad \ldots(2.6.3)$$

an $r_i \times r_i$ matrix where r_i is the multiplicity of the eigenvalue λ_i. The diagonal matrix is a special case of Jordan matrix with each $r_i = 1$.

3) Companion Form : This is given as

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 1 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & 0 \\ 0 & & & & & 0 & 1 \\ -a_0 & -a_1 & \cdot & \cdot & \cdot & \cdot & -a_{n-1} \end{bmatrix} \qquad \ldots(2.6.4)$$

When the system evolution matrix is transformed to this form, its characteristic equation can be written just by inspection as

$$a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + s^n = 0$$

If the corresponding $\bar{B}$ matrix in a scalar system is

$$\bar{B}^T = [0, 0, \ldots 1] \qquad \ldots(2.6.5)$$

then the system is known to be controllable. Hence the above form 2.6.4 is said to be scalar controllable companion form. Secondly corresponding to $\bar{A}$ and $\bar{B}$ of 2.6.4 and 2.6.5, the state variables are x, $\dot{x}$, $\ddot{x}$, ... etc. Hence it is also called the phase variable form.

A multivariable control system has the following controllable companion forms for $\bar{A}$ and $\bar{B}$ matrices.

$$\bar{A} = \left[\begin{array}{cccc:ccc:c:ccc}
 & A_1 & & & 0 & . & 0 & & 0 & . & 0 \\
 & & & & . & & . & & . & & . \\
 & & & & 0 & . & 0 & & 0 & . & 0 \\
x & x & x & x & x & x & x & & x & x & x \\ \hdashline
0 & . & . & 0 & & & & & 0 & . & 0 \\
. & . & . & . & & A_2 & & & . & . & . \\
x & x & x & x & x & x & x & & x & x & x \\ \hdashline
 & & & & & & & & & & \\ \hdashline
 & & & & & & & & & A_m & \\
x & \cdots & \cdots & x & x & \cdots & \cdots & \cdots\ x & x & x & x
\end{array}\right] \qquad \ldots(2.6.6)$$

where each diagonal block A_i has the form 2.6.4 and is a sq. matrix of degree d_i such that

$$\sum_{i=1}^{m} d_i = n \qquad \ldots(2.6.7)$$

All the $\sigma_k^{\,th}$ rows defined below, of $\bar{A}$ are nonzero rows and the remaining rows of the off diagonal blocks are zero. Here

$$\sigma_k = \sum_{i=1}^{k} d_i = \text{for } k = 1 \text{ to } m \qquad \ldots(2.6.8)$$

so that $\sigma_m = n$

The corresponding form of $\bar{B}$ is

$$\bar{B} = \begin{bmatrix} 0 & \cdots & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x & \cdots & & x \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & x & \cdots & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad \ldots(2.6.9)$$

where again the only nonzero rows are the σ_k^{th} rows for $k = 1$ to m. These rows of $\bar{A}$ and $\bar{B}$ of 2.6.6 and 2.6.9 are called the significant rows.

If the evolution matrix $\bar{A}$ and observation matrix $\bar{C}$ of a system are in the form of transpose of $\bar{A}$ of 2.6.4 or 2.6.6 and $\bar{B}$ of 2.6.5 or 2.6.9 respectively, then the corresponding system is observable. Hence these forms (i.e. transposes) are called the observable companion forms. Note that controllability depends on the pair (A, B), whereas observability depends on the pair (A, C) as will be cdlasified later.

The important canonical forms of polynomial matrices used in the D.O. and T.M. representations are :

a) Diagonal Form, Smith McMillan Form

b) Upper right triangular form

c) Lower left triangular form

a) A polynomial square matrix (R(s) is equivalent to a diagonal matrix.

$$\text{diag}\left[r_1(s),\ r_2(s)\ \ldots\ r_q(s),\ 0\ \ldots\ 0\right] \qquad \ldots(2.6.10\,a)$$

if its rank is q.

If R(s) is of dimension p x m, then it can be converted to equivalent forms :

$$\left[\text{diag.}\left[r_1(s)\ r_2(s)\ \dots\ r_q(s),\ 0\dots 0 \right] \vdots\ 0_{p,m-p} \right] \qquad \dots(2.6.10b)$$

if p < m, and

$$\begin{bmatrix} \text{diag}\left[r_1(s)\ \ r_2(s)\ \dots\ r_q(s),\ 0\ ..\ 0 \right] \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 0_{p-m,\ m} \end{bmatrix} \qquad \dots(2.6.10c)$$

if p > m.

In each of the forms (2.6.10) a, b, c, the polynomials $r_i(s)$ are monic. Any $r_i(s)$ devides $r_{i+1}(s)$. This form of a p x m polynomial matrix R(s) is called the Smith form, $R_s(s)$ say. The polynomials $r_i(s)$ are the invariant polynomials of R(s) and the rank of R(s) = q = the number of nonzero invariant polynomials.

A transfer matrix T(s) can be converted to the Smith McMillan form. It can be written first as

$$T(s) = \frac{R(s)}{p(s)} \qquad \dots(2.6.11)$$

where p(s) is the least common multiple of the denominators of the elements of T(s) and is monic. Then R(s) can be converted to equivalent Smith form $R_s(s)$. The Smith McMillan form $T_{sm}(s)$ of T(s) can be obtained by cancelling the common factors in $R_s(s)/p(s)$. The poles and zeros of a transfer matrix are the poles and zeros of the diagonal elements of

its Smith McMillan form.

The diagonal form is useful for decoupling. In the case of square transfer matrix it makes inversion easier. It reveals the zeros and poles of the transfer matrix.

b) Upper right triangular form is given by

$$
\begin{array}{ll}
R(s) = & \left[\begin{array}{ccccc}
x & \cdots & x & \cdots & x \\
0 & x & \cdots & \cdots & x \\
\cdot & \cdots & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdots & \cdot \\
0 & \cdots & \cdots & 0 & x \\
\cdot & \cdots & \cdots & \cdots & \cdot \\
0 & \cdots & \cdots & \cdots & 0 \\
\cdot & \cdots & \cdots & \cdots & \cdot \\
0 & \cdots & \cdots & \cdots & 0
\end{array}\right]
\begin{array}{l}
m \times m \\ \\ \\ \\ \\ \\ \\ (p-m) \times m \\ \\
\end{array} \\
p > m &
\end{array}
\qquad \ldots(2.6.12)
$$

or

$$
\begin{array}{ll}
R(s) = & \left[\begin{array}{cccccccc}
x & \cdot & x & \cdots & x & \cdots & \cdots & x \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\
\cdot & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot \\
0 & \cdots & 0 & 0 & x & \cdots & \cdots & x
\end{array}\right] \\
p < m & \quad p \times p \qquad\qquad\qquad p \times (m-p)
\end{array}
$$

c) The lower left triangular forms are transpose of 2.6.11 and 2.6.12. These triangular forms are very useful in frequency domain analysis and design as will be clarified later.

The upper right triangular forms can be obtained by row operations and the lower left triangular forms can be obtained by column operations as explained later in Chapter-5.

2.7 Proper Polynomial and Transfer Matrices

R(s) of 2.4.2 is called a proper polynomial matrix if R_n, the matrix coefficient of the highest order term, is nonsingular.

The degree of the highest degree element of any column of a polynomial matrix is called the column degree of that column. Thus a p x m polynomial matrix R(s) can be written also as,

$$R(s) = \gamma_c \begin{bmatrix} s^{d_{*1}} & \cdot & \cdot & \cdot \\ \cdot & s^{d_{*2}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & s^{d_{*m}} \end{bmatrix} + \gamma_{c1} \begin{bmatrix} s^{d_{*1}-1} & \cdot & \cdot & \cdot \\ \cdot & s^{d_{*2}-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & s^{d_{*m}-1} \end{bmatrix}$$

$$+ \ldots \qquad \ldots(2.7.1)$$

where $d_{*1}, d_{*2} \ldots d_{*m}$ are the column degrees of the m columns of R(s). R(s) is called column proper if γ_c is nonsingular.

Similarly any R(s) can be written in the form;

$$R(s) = \begin{bmatrix} s^{d_{1*}} & \cdot & \cdot & \cdot \\ \cdot & s^{d_{2*}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & s^{d_{p*}} \end{bmatrix} \gamma_r + \begin{bmatrix} s^{d_{1*}-1} & \cdot & \cdot & \cdot \\ \cdot & s^{d_{2*}-1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & s^{d_{p*}-1} \end{bmatrix} \gamma_{r1}$$

$$+ \ldots\ldots \qquad \ldots(2.7.2)$$

where $d_{1*}\ d_{2*} \ldots d_{p*}$ are the degrees of the p rows of R(s). Then R(s) is called row proper if γ_r is nonsingular. A transfer matrix T(s) is said to be proper if in all its

transfer function elements, the degree of **numerator is smaller** than or equal to the degree of the denominator. Generally the transfer matrices of practical systems are proper.

Any proper transfer matrix T(s), expressed as,

$$T(s) = R(s)\, P^{-1}(s) \qquad \ldots(2.7.3)$$

where R(s) and P(s) are polynomial matrices, following condition is satisfied.

$$d\,[P(s)]_{*j} \geq d\,[R(s)]_{*j} \qquad \ldots(2.7.4)$$

where d denotes degree and *j denotes j^{th} column.

If for any polynomial matrices R(s) and P(s),

$$d\,[P(s)]_{*j} < d\,[R(s)]_{*j} \qquad \ldots(2.7.5)$$

then the transfer matrix $R(s)\, P^{-1}(s)$ cannot be a proper transfer matrix. However the condition 2.7.4 may be satisfied by a proper or a non proper transfer matrix.

These facts may be proved by the reader as an exercise.

Exercise 2 : -

(2.1) Identify the form of representation for the following examples :

i) $$3\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x + \frac{dy}{dt} + 2y = \sin t$$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y + 7x = 2 + \cos t$$

ii) $$\frac{dx_1}{dt} = -5x_1 - 2x_2 + u_1$$

$$\frac{dx_2}{dt} = -x_1 - 3x_3 + u_2$$

$$\frac{dx_3}{dt} = -2x_1 - x_2 - x_3$$

iii) $$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

iv) $$y(s) = \begin{bmatrix} \dfrac{2s+3}{s^2+s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{s+2}, & s \end{bmatrix} u(s)$$

v) $$\begin{bmatrix} 3D+2 & D+1 \\ D & 1 \end{bmatrix} z(t) = \begin{bmatrix} D & 1 \\ 1 & 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} D+1 & D \\ 0 & 2 \end{bmatrix} z(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} u(t)$$

(2.2) Calculate the transfer matrices for each of the systems of Example (1).

(2.3) Are the following state vectors equivalent to one another? What are the transformation matrices for (b) and (c) w.r.t. (a) ?

(a) $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ (b) $\bar{x} = \begin{bmatrix} 4\, x_1 + 2\, x_2 + x_3 + x_4 \\ 2\, x_2 + x_4 \\ x_1 + 2\, x_3 \\ x_4 \end{bmatrix}$

(c) $\hat{x} = \begin{bmatrix} 4\, x_1 + 2\, x_2 + x_3 + x_4 \\ 2\, x_2 + x_4 \\ 2\, x_1 + x_2 \\ -2\, x_2 + x_3 \end{bmatrix}$

(2.4) If a system is described by the state space quadruple (A, B, C, E) with x of 3(a) as the state vector, and the quadruple ($\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{E}$) of the equivalent system with equivalent state vector $\bar{x}$ of 3(b) and if

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -1 & -3 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \bar{C} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \bar{E} = 0$$

Calculate (A, B, C, E). Obtain the characteristic polynomial.

(2.5) Identify whether the following polynomial matrices are proper, column proper, row proper.

(i) $\begin{bmatrix} s^2+s+1 & 2s+1 \\ s & s^2+2 \end{bmatrix}$ (ii) $\begin{bmatrix} s+1 & s+2 \\ 1 & 3s^2+1 \end{bmatrix}$

(iii) $\begin{bmatrix} s+1 & s^2+2s+2 \\ 2 & s^2 \end{bmatrix}$

(2.6) (a) Express the transfer matrix of example (2.1)-(iv) as a sum of a strictly proper transfer matrix and a polynomial matrix.

(b) Express the strictly proper part T(s) of (a) in the form $R(s)\,p^{-1}(s)$ where p(s) is a monic polynomial.

(2.7) Is the converse of property (i) on page 2.7 true? Verify your answer for

$$A = \begin{bmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{bmatrix} \text{ and } \bar{A} = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix}$$

+ + +

CHAPTER - 3

SOLUTION

3.1 Introduction :

Finding the solution of the system is to evaluate the set of system variables as functions of time, given inputs and n terminal conditions where n is the order of the system. Just as in the case of representation solution can be obtained by using either the time domain techniques or frequency domain techniques.

In time domain a system is represented by a set of differential equations. The solution of the system is the solution of these differential equations. The solution is easier if the n terminal conditions specified are the initial conditions. The set of differential equations can be solved in (i) time domain or (ii) in frequency domain, using Laplace transformation. The time domain solution is easy for 1^{st} or second order differential equations. Hence system represented by S.S. method can be easily solved by time domain methods. For systems of highest order use of Laplace transform is more convenient. Hence the system represented in D.O. form can conveniently be solved using Laplace transformation. The T.M. representation itself uses Laplace transformation. Naturally frequency domain technique solution will be convenient in this

case. Laplace transformation is useful in solving the system in its S.S. form also.

Thus in this chapter the time domain method of solution for S.S. representation and frequency domain technique of solution of system in all the three types of representations are considered.

Once we get the system solution the system response, i.e. the output $y(t)$, can be obtained using the relations 2.2.1 or 2.3.1 as the case may be. Thus the system response depends on (i) initial conditions and (ii) inputs. So we can define zero state response and zero input response as the responses for (i) equal to zero and (ii) equal to zero respectively.

As S.S. representation is a very explicit form of representation we can define many other terms related to solution.

The solution of S.S. representation of the system in time domain is explained in two parts :

i) The solution of homogeneous state space equations, which corresponds to zero input solution.

ii) The solution of nonhomogeneous state space equations.

3.2 Solution of Homogeneous State Equations

Assuming that the initial state $x(t_0)$ is given and $u(t)$ is given for time interval, $t_0 \leq t \leq t_1$, the solution $x(t)$ over the same time interval can be determined for the state equations of 2.3.1. When the forcing function vector $u(t)$ is

zero, the state equations are homogeneous i.e.,

$$\dot{x} = Ax \qquad \ldots(3.2.1)$$

With analogy to a first order differential equation in a scalar variable x, let us assume the solution of 3.2.1 to be of the form

$$x(t) = e^{At}\, x(o) \qquad \ldots(3.2.2)$$

where e^{At} denotes the infinite series

$$e^{At} = I + At + \frac{A^2\, t^2}{2!} + \ldots \frac{A^k t^k}{k!} + \ldots \qquad \ldots(3.2.3)$$

and x(o) is the initial state at $t_o = 0$. Differentiating 3.2.2

$$\dot{x} = \left[A + \frac{2A^2 t}{2!} + \ldots \frac{k\, A^k\, t^{k-1}}{k!} \right] x(o)$$

$$= A \left[I + A\,t + \frac{A^2 t^2}{2!} + \ldots \right] x(o)$$

$$= Ae^{At}\, x(o)$$

$$= Ax \qquad \ldots(3.2.4)$$

Thus 3.2.2 satisfies 3.2.1 and hence is its solution. As the state x(t) depends on the matrix e^{At} and on the given initial condition, the matrix e^{At} is called the state transition matrix. It is also denoted by $\phi(t)$.

If initial time is t_o instead of zero, the state transition matrix changing the state from $x(t_o)$ to $x(t)$ is

$$\phi(t_1\; t_o) = e^{A(t_1 - t_o)} \qquad \ldots(3.2.5)$$

3.3 The Solution Space :

The solution $x(t)$ of 3.2.2 is an n dimensional vector. A set of all such possible solutions, for various $x(t_o)$, constitute the n dimensional state space. Note that $x(t_o)$, the initial state also belongs to this state space. The standard basis of any n dimensional space is given by the n unit vectors

$$
\begin{aligned}
e_1 &= [1 \quad 0 \; . \; . \; . \; . \; 0]^T \\
e_2 &= [0 \quad 1 \quad 0 \; . \; . \quad 0]^T \\
&. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\
e_n &= [0 \quad 0 \; . \; . \; . \; 0 \; 1]^T
\end{aligned}
\qquad \text{...(3.3.1)}
$$

Any set of n linearly independent vectors can form the basis of an n dimensional state space. The set of n linearly independent vectors of the specific form defined by 3.3.1 is called the standard basis. Let $x_1(t)$, $x_2(t)$ .... $x_n(t)$ be n dimensional vectors representing the solution for the initial condition vectors given by e_1, e_2 ... e_n of 3.3.1, respectively i.e.

$$
\begin{aligned}
x_1(t) &= \phi(t)\, e_1 \\
x_2(t) &= \phi(t)\, e_2 \\
&. \; . \; . \; . \; . \; . \; . \; . \\
x_n(t) &= \phi(t)\, e_n
\end{aligned}
\qquad \text{...(3.3.2)}
$$

The set of vectors $x_1(t)$, $x_2(t)$ ... $x_n(t)$ is linearly independent. This can be proved by contradiction as follows.

Let us assume that $x_1(t)$, $x_2(t)$ ... $x_n(t)$ are linearly dependent and let a set of n constants b_1 ... b_n, not all zero,

satisfy

$$b_1 x_1(t) + b_2 x_2(t) + \dots b_n x_n(t) = 0 \qquad \dots(3.3.3)$$

From 3.3.2 it can be written as

$$\phi(t) \left[b_1 e_1 + b_2 e_2 \dots + b_n e_n \right] = 0 \qquad \dots(3.3.4)$$

This to be true for all t, requires that

$$b_1 e_1 + b_2 e_2 + \dots b_n e_n = 0 \qquad \dots(3.3.5)$$

But this contradicts the fact that $e_1, e_2 \dots e_n$ are linearly independent. The $x_1(t), x_2(t) \dots x_n(t)$ are proved to be linearly independent. A matrix $x_f(t)$, formed by $x_1(t), x_2(t) \dots x_n(t)$ as its columns is obviously equal to $\phi(t)$ and is called the fundamental matrix of the homogeneous state equations 3.2.1. Obviously $x_f(t)$ is nonsingular. The set $x_1(t), x_2(t) \dots x_n(t)$ is a basis of the state space.

3.4 Complete Solution of the State Equations :

Let the matrix state equation be premultiplied by e^{-At} to give

$$e^{-At} \dot{x} = e^{-At} Ax + e^{-At} Bu \qquad \dots(3.4.1)$$

i.e. $e^{-At} \dot{x} - e^{-At} Ax = e^{-At} Bu$

i.e. $\dfrac{d}{dt} \left[e^{-At} x \right] = e^{-At} Bu$

$\therefore \quad e^{-At} x = e^{-At_o} x(t_o) + \int_{t_o}^{t} e^{-A\tau} Bu(\tau)\, d\tau$

$$\therefore \quad x = e^{A(t-t_o)} x(t_o) + \int_{t_o}^{t} e^{A(t-\tau)} Bu(\tau)\, d\tau \qquad \dots(3.4.2)$$

3.5 System Response :

The system response defined by equation 2.2.1 can be written by substituting for x from 3.4.2, as

$$y(t) = C\,e^{A(t-t_o)}\,x(t_o) + C\int_{t_o}^{t} e^{A(t-\tau)}\,Bu(\tau)\,d\tau + Eu(t) \qquad \ldots(3.5.1)$$

Clearly the part $Ce^{A(t-t_o)}\,x(t_o)$ can be called the zero input response and the remaining two terms give the zero state response.

Another matrix called impulse response matrix is important as a link between the state space and the transfer matrix representations. Let $y_1(t)$, $y_2(t)$ .. $y_m(t)$ be the output vectors corresponding to input vectors $u_1(t)$, $u_2(t)$ ... $u_m(t)$ where

$$u_i(t) = [0\ \ldots\ 0\ \delta(t)\ 0\ \ldots\ 0)]^T \qquad \ldots(3.5.2)$$

where $\delta(t)$ (= a unit impulse at t_o=0) is the i^{th} element of $u_i(t)$. Then the impulse response matrix $T(t)$ is defined as the p x m matrix having $y_1(t)$, $y_2(t)$ ... $y_m(t)$ as its columns. Any $y_i(t)$ is given by

$$y_i(t) = C\,e^{At}\,B_{*i} + E_{*1}\,\delta(t) \qquad \ldots(3.5.3)$$

Obviously the Laplace transform of $T(t)$ gives the transfer matrix $T(s)$ of the system.

3.6 Determination of e^{At} :

Thus for getting solution for a given state space representation, it is first necessary to get e^{At} for the given A. This can be done by many methods. Some of them are listed below :

i) Computation of the exponential series 3.2.3. This is not very much desirable as it is an infinite series though generally converging after a reasonable number of terms.

ii) The sylvester theorem gives e^{At} as

$$e^{At} = \sum_{r=1}^{n} \mathrm{Exp}\ \lambda_r t \prod_{s \neq r} \frac{A - \lambda_s I}{\lambda_r - \lambda_s} \qquad \ldots(3.6.1)$$

for A having distinct eigenvalues λ_r. The series is limited to n terms only. The evaluation of eigenvalues has to be done first.

If A is converted into the diagonal form

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{bmatrix} \qquad \ldots(3.6.2)$$

by equivalent transformation, then by substituting this in 3.6.1 it can be shown that

$$e^{\bar{A}t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdots & \cdot \\ 0 & \cdots & \cdots & \cdot & e^{\lambda_n t} \end{bmatrix} \qquad \ldots(3.6.3)$$

iii) By using equivalent transformation any A with distinct eigenvalues can be converted into $\bar{A}$ of diagonal form 3.6.2. Then using the relation

$$e^{At} = Q^{-1}\, e^{\bar{A}t}\, Q \qquad \ldots(3.6.4)$$

and using the relation (3.6.3), e^{At} can be determined. By giving initial conditions of (3.3.1), the state vectors $x_1(t) \ldots x_n(t)$ of (3.3.2) can be obtained. Hence :

$$\phi(t) = e^{At} = x_f(t) \qquad \ldots(3.6.5)$$

can be obtained by simulation of the state space equations $\dot{x} = Ax$ with initial conditions (3.3.1) applied in sequence.

v) Using frequency domain technique i.e. writing the Laplace transform of the state equations,

$$Sx - x(o) = Ax + Bu$$

$$\therefore \quad (sI-A)x = x(o) + Bu$$

$$\therefore \quad x = L^{-1}\left[(sI-A)^{-1}\,x(o) + (sI-A)^{-1}\,Bu\right] \qquad \ldots(3.6.6)$$

comparing with 3.4.2,

$$e^{At} = L^{-1}\,(sI-A)^{-1} \qquad \ldots(3.6.7)$$

Note that 3.6.6 gives the solution of state equations by Laplace transform method. The response can be obtained by substituting x(s) in

$$y(s) = C\,x(s) + E\,u(s) \qquad \ldots(3.6.8)$$

and taking inverse Laplace of y(s).

It can also be obtained directly if the transfer matrix T(s) is known and u(s) is given.

It can be seen that the vector of system variables $z(t)$ and outputs $y(t)$ in D.O. form can also be obtained easily by Laplace transform method, as follows.

$$P(s)\ z(s) = Q(s)\ u(s)$$

$$y(s) = R(s)\ z(s) + W(s)\ u(s)$$

i.e. $$z(s) = P^{-1}(s)\ Q(s)\ u(s) \qquad \ldots(3.6.9)$$

$$y(s) = R(s)\ P^{-1}(s)\ Q(s)\ u(s) + W(s)\ u(s) \qquad \ldots(3.6.10)$$

Exercise 3 :-

(3.1) In the following circuit choose the current in L and voltage across C as the state variables and voltages across R_o and C as outputs. Write the state equations. If $R = R_o = C = L = 2$ in consistent units, solve the circuit to obtain the state vector and the output vector at time t. Assume zero initial conditions and $V_1(t) = u(t)$ and $V_2(t) = 2\,u(t)$

(3.2) Given $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Determine e^{At} by infinite series approximated by first three terms. Compare the result with those obtained by applying Sylvester expansion.

(3.3) Calculate the fundamental matrix $X_f(t)$ of the state equations of problem 1.

(3.4) Diagonalise the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{bmatrix}$$

Hence obtain $\phi\,(t)$.

(3.5) Calculate e^{At} for the system of problem (2.1 - ii), using Laplace transformation. Hence calculate the output for initial conditions

$$x(o) = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix}^T$$

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CHAPTER - 4

CONTROLLABILITY

4.1 Introduction :

Consideration of system stability is the most important aspect of analysis and design in classical theory. In modern control system theory, some more concepts have to be introduced as equally important. These are controllability, observability and reproducibility. These properties could be recognized because of the more explicit description of the control system in the S.S. form. This and the following chapters explain these properties with reference to the three standard forms of representation, using the one most suitable for each of them.

4.2 Definition of Controllability :

The purpose of any control system is fully served only if the given state of the system can be transformed to a desired state by applying any possible input. A system with this property is said to be controllable. Its definition can be stated as :

A system is said to be controllable if and only if the given initial state $x(t_o)$ of the system can be transferred to a desired state $x(t_1)$ at time t_1 by an input $u(t)$ $t_o \leq t \leq t_1$ from the set of possible inputs.

If some of the state variables are not controllable in the above sense the system is said to be uncontrollable. If all

the output variables are controllable, the system is said to be output controllable. An output controllable system need not necessarily be completely state controllable. That is, output controllability is a less restrictive specification on the system than (complete) state controllability.

4.3 Controllability Criteria :

From the definition of controllability, the criterion is that a u(t) must exist, for given $x(t_o)$ and desired $x(t_1)$, such that :

$$x(t_1) = e^{A(t_1-t_o)}\, x(t_o) + \int_{t_o}^{t_1} e^{A(t_1-\tau)}\, Bu(\tau)d\tau \qquad \ldots(4.3.1)$$

i.e. $$x(t_1) - e^{At_1}\, x(o) = \int_{o}^{t_1} e^{A(t_1-\tau)}\, B\, u(\tau)\, d\tau \qquad \ldots(4.3.2)$$

for $t_o = 0$

i.e. $$e^{-At_1}\, x(t_1) - x(o) = \int_{o}^{t_1} e^{-A\tau}\, B\, u(\tau)\, d\tau \qquad \ldots(4.3.3)$$

$$e^{-A\tau} = \sum_{r=o}^{n-1} K_r(\tau)\, A^r \qquad \ldots(4.3.4)$$

by Calay Hamilton theorem (a-2) and

$$B\, u(\tau) = \sum_{s=1}^{m} b_s\, u_s\, (\tau) \qquad \ldots(4.3.5)$$

where b_s is the s^{th} column of B and u_s is the s^{th} element of u. Substituting 4.3.4 and 4.3.5 in 4.3.3,

$$e^{-At_1}\, x(t_1) - x(o)$$

$$= \int_o^{t_1} \sum_{r=o}^{n-1} K_r(\tau)\, A^r \sum_{s=1}^{m} b_s\, u_s(\tau)\, d\tau$$

$$= \sum_{r=o}^{n-1} \sum_{s=1}^{m} A^r\, b_s \int_o^{t_1} K^r(\tau)\, u_s(\tau)\, d\tau \qquad \ldots(4.3.6)$$

The L.H.S. of 4.3.6 is any arbitrary vector, expressed as the linear combination of the columns $[A^r\, b_s]_{\substack{r=o..(n-1)\\ s=1..m}}$, the integral representing the multiplying constants. Hence at least n columns of $[A^r\, b_s]_{\substack{r=o\,..(n-1)\\ s=1\,..m}}$ i.e. of

$$M_c = [B\ AB\ A^2B\ \ldots\ A^{n-1}B\ ] \qquad \ldots(4.3.7)$$

must represent the n dimensional basis of the vector space X of all n dimensional vectors x. This is equivalent to saying that the necessary criterion for controllability is that the rank of the n x nm controllability matrix M_c should be n.

Let the matrix M_c have its rank less than n, corresponding to an uncontrollable system. Then for some constant row vector $\alpha = [\alpha_1\ \alpha_2\ \ldots\ \alpha_n]$ we can write

$$\alpha\, [B\ AB\ \ldots\ A^{n-1}B\ ] = [0] \qquad \ldots(4.3.9)$$

Then the nm elements of the resulting l.h.s. matrix are each zero. Writing expression for the $(m\ r + i)^{th}$ element,

$$\alpha_1 [A^r\, b_i]_1 + \alpha_2 [A^r\, b_i]_2 + \ldots + \alpha_n [A^r\, b_i]_n = 0 \ldots(4.3.10)$$

where $[A^r\, b_i]_k$ is the k^{th} element of the column vector $[A^r\, b_i]$

$$\therefore \quad K_r(t)\left[\alpha_1 \left[A^r b_i\right]_1 + \alpha_2 \left[A^r b_i\right]_2 + \dots + \alpha_n \left[A^r b_i\right]_n\right] = 0$$

for $r = 0, 1, 2, \dots n-1$ and $i = 1, 2, \dots m$ ...(4.3.11)

$$\therefore \quad \alpha \sum_{r=0}^{n-1} K_r(t)\, A^r B = [0] \qquad \dots(4.3.12)$$

The functions $K_r(t)$ in (4.3.11) and (4.3.12) can be chosen such that 4.3.12 can be written as,

$$\therefore \quad \alpha\, e^{At} B = [0] \qquad \dots(4.3.13)$$

This shows that if the n rows of M_c are linearly dependent, the n rows of e^{At} B are also linearly dependent. If M_c is of rank n, then any nonzero α satisfying 4.3.9 does not exit. So any α satisfying 4.3.13 also does not exist. So n rows of e^{At} B are linearly independent over R, the real field.

If for some nonzero α, 4.3.13 is true we can write

$$\int_{t_0}^{t_1} \alpha\, e^{-A\tau} B \left[\alpha\, e^{-A\tau} B\right]^T d\tau = 0 \qquad \dots(4.3.14)$$

i.e. $$\alpha \left[\int_{t_0}^{t_1} e^{-A\tau} B\, B^T e^{-A^T\tau}\, d\tau\right] \alpha^T = 0 \qquad \dots(4.3.15)$$

For nonzero α, 4.3.15 means the n x n matrix in the brackets denoted by,

$$\underline{W(t_0,t_1)} = \int_{t_0}^{t_1} e^{-A\tau} B\, B^T e^{-A^T\tau}\, d\tau \qquad \dots(4.3.16)$$

is singular.

If no such nonzero α can be defined, $W(t_o,t_1)$ is nonsingular. In that case a u(t) defined by

$$u(t) = B^T e^{-A^T t} \cdot W^{-1}(t_o,t_1) \left[e^{-At_1} x(t_1) - e^{At_o} x(t_o) \right]$$

...(4.3.17)

can take the state from $x(t_o)$ to $x(t_1)$ satisfying the requirement for controllability.

The above derivation proves that the following three equivalent criteria are necessary and sufficient conditions for a system to be controllable

i) $[B, AB \ldots A^{n-1} B] = n$

ii) n rows of e^{At} B are linearly independent over the field of real numbers

iii) $W(t_o,t_1)$ is nonsingular ...(4.3.18)

4.4 Controllable Companion Transformation :

The suitable forms of evolution matrix A for revealing the controllability of given scalar or multivariable control system are already given by 2.6.4 and 2.6.6. As seen in the previous section controllability depends on the pair (A, B). Hence there must be a controllable form of B corresponding to the controllable companion form of A. In this section, the transformation matrix Q, for conversion of any pair A, B into its equivalent controllable companion form, is derived. Both scalar and multivariable cases are considered.

1) Scalar Controllable Companion Form :

The required n x n transformation matrix Q is such that

$$Q A = \bar{A} Q$$

...(4.4.1)

where

$$\underset{n\times n}{\bar{A}} = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & 1 \\ -a_0 & -a_1 & \cdots & \cdot & -a_{n-1} \end{bmatrix}$$

and $\underset{n\times 1}{\bar{B}} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$ ...(4.4.2)

i.e. $$\begin{bmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{bmatrix} A = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & 1 \\ -a_0 & \cdot & \cdots & \cdots & -a_{n-1} \end{bmatrix} Q$$ ...(4.4.3)

where q_i denotes i^{th} row of Q.

The first row of l.h.s. of 4.4.3

$= q_1 A$

$=$ the first row of r.h.s.of 4.4.3

$=$ second row of $Q = q_2$

i.e. $q_1 A = q_2$

Similarly $q_2 A = q_3$

i.e. $q_3 = q_1 A^2$

Thus $q_2 = q_1 A$

$q_3 = q_1 A^2$

$\cdots\cdots$

$q_n = q_1 A^{n-1}$ ...(4.4.4)

Thus, once the first row of Q is determined all other rows can be derived using 4.4.4. The first row can be determined using standard form of $\bar{B}$,

$$\bar{B} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} = Q\,B = \begin{bmatrix} q_1 \\ q_1\,A \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_1\,A^{n-1} \end{bmatrix} B \qquad \ldots(4.4.5)$$

$$\therefore \quad \bar{B}^T = [\; q_1B \quad q_1AB \quad q_1A^2B \;\ldots\ldots\; q_1A^{n-1}\,B\,]$$

$$= q_1\,[B \quad AB \quad A^2B \;\ldots\; A^{n-1}\,B]$$

$$= q_1 M_c$$

$$\therefore \quad [0 \;.\;.\;.\;.\;.\; 1]\; M_c^{-1} = q_1$$

i.e. q_1 is the last row of M_c^{-1} ... (4.4.6)

Thus whenever a system is controllable, M_c is invertible and q_1 is defined as above. Hence, conversely whenever A, B can be represented in the above form, the pair is controllable.

2) Multivariable Controllable Companion Form :

This form is specially useful for the linear state variable feedback technique of compensation. Any controllable pair $(\bar{A}, \bar{B})$ can be transformed to the following multivariable controllable companion form suggested first by Luenberger (L4)

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m_1} & A_{m_2} & & A_{mm} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}$$

where, A_{ii} is of dimension $d_i \times d_i$

A_{ij} is of dimension $d_i \times d_j$

B_i is of dimension $d_i \times m$

such that $\sum_{i=1}^{m} d_i = n$

and they have the following forms :

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \\ x & \cdot & \cdot & \cdot & \cdot & x \end{bmatrix} \qquad A_{ij} = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ x & \cdot & \cdot & \cdot & \cdot & x \end{bmatrix}$$

and

$$B_i = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & 0 & \cdots & 0 \\ 0 & & 0 & 1 & X & \cdots & X \end{bmatrix} \qquad \ldots(4.4.8)$$

where the first i-1 columns are zero. The required transformation matrix can be derived from the equivalence relations,

$$A\,Q = Q\,\overline{A}$$

and

$$B = Q\,\overline{B} \qquad \ldots(4.4.9)$$

as derived in 2.5.5. Writing 4.4.9 in terms of partitioned matrices,

$$\begin{bmatrix} q(1) \\ q(2) \\ \vdots \\ q(m) \end{bmatrix} \bar{A} = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{1m} & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} q(1) \\ \vdots \\ q(m) \end{bmatrix} \quad \ldots(4.4.10)$$

where $q(i)$ is d_i x n matrix.

First row of 4.4.10 gives

$$q(1)\,\bar{A} = A_{11}\,q(1) + A_{12}\,q(2) + \ldots A_{1m}\,q(m) \quad \ldots(4.4.11)$$

Following the notation, T_{i*} and T_{*j} are the i^{th} row and j^{th} column respectively of a matrix T, 4.4.11 can be expanded as,

$$\begin{bmatrix} q(1)_{1*} \\ q(1)_{2*} \\ \vdots \\ q(1)_{d_1*} \end{bmatrix} \bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdots & & 0 \\ \cdot & \cdot & \cdot & \cdots & 0 & 1 \\ x & \cdot & \cdot & \cdots & x & x \end{bmatrix} \begin{bmatrix} q(1)_{1*} \\ \vdots \\ q(1)_{d_1*} \end{bmatrix}$$

$$\begin{bmatrix} 0 & \cdots & 0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ 0 & \cdots & 0 \\ x & \cdots & x \end{bmatrix} \begin{bmatrix} q(2)_{1*} \\ \vdots \\ q(m)_{d_2*} \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ 0 & \cdots & \cdot \\ x & \cdots & x \end{bmatrix} \begin{bmatrix} q(m)_{1*} \\ \vdots \\ q(m)_{d_m*} \end{bmatrix}$$

$$\ldots(4.4.12)$$

The first d_1-1 rows of 4.4.12 can be written as,

$$q(1)_{1*}\,\bar{A} = q(1)_{2*} = q(1)_{1*}\,\bar{A}$$

$$q(1)_{2*}\,\bar{A} = q(1)_{3*} = q(1)_{1*}\,\bar{A}^2$$

$$\cdots\cdots\cdots$$

$$q(1)_{(d_1-1)_*}\bar{A} = q(1)_{d_1*} = q(1)_{1*}\,\bar{A}^{d_1-1} \qquad \ldots(4.4.13)$$

Thus the assumed form of A gives the specific relation 4.4.13 between the first row and any other row of q(1). The first row of q(1) can be chosen so as to obtain the required form of B. Thus we require, for the first d_1 rows of B.

$$\begin{bmatrix} q(1)_{1*} \\ \vdots \\ \vdots \\ q(1)_{d_1*} \end{bmatrix} \bar{B} = \begin{bmatrix} q(1)_{1*} \\ q(1)_{1*}\bar{A} \\ \vdots \\ \vdots \\ q(1)_{1*}\bar{A}^{d_1-1} \end{bmatrix} \begin{bmatrix} \bar{B}_{*1} & \cdots & \bar{B}_{*m} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & X & \cdots & X \end{bmatrix}$$

or, in general

$$\begin{bmatrix} q(i)_{1*} \\ \vdots \\ q(i)_{d_i*} \end{bmatrix} \bar{B} = \begin{bmatrix} q(i)_{1*} \\ q(i)_{1*}\bar{A} \\ \vdots \\ q(i)_{1*}\bar{A}^{d_i-1} \end{bmatrix} \bar{B} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & X \cdots & X \end{bmatrix} \qquad \ldots(4.4.14)$$

The general relations similar to 4.4.13 can be obtained for the rows of q(i) by expanding 4.4.10 for i = 2,3,... m. Following choice of Q given by Luenberger (L-1) satisfies all these relations. A matrix L is defined first, on which first row of each q(i), i = 1, 2, ..., m depends.

$$L = \left[\bar{B}_{*1} \; \bar{A}\,\bar{B}_{*1} \; \bar{A}^2\,\bar{B}_{*1} \ldots \bar{A}^{d_1-1}\,\bar{B}_{*2} \; \bar{A}\,\bar{B}_{*2} \ldots \bar{B}_{*m} \ldots \bar{A}^{d_m-1}\,\bar{B}_{*m} \right] \qquad \ldots(4.4.15)$$

The columns of L are obtained from the controllability matrix M_c defined by 4.3.7. Going from left to right any columns of M_c, linearly dependent on previous columns are eliminated and the procedure is continued till a set of n linearly independent columns is formed. This set is then rearranged as shown in 4.4.15 to form L. The numbers $d_1, d_2 \ldots d_m$ in the L, thus obtained, are called the controllability indices. The maximum of these d_i's i = 1,2,... m is called the controllability index of the S.S. system and is denoted by μ.

Let, $$\sigma_k = \sum_{i=1}^{k} d_i \quad \text{for } k = 1, 2, \ldots m \qquad \ldots(4.4.16)$$

Then the first rows of the m blocks of Q, namely $q(1)_{1*} \ldots q(m)_{1*}$ are chosen to be equal to the σ_kth rows of L^{-1} for k = 1,2,..,m. It is shown below that this choice of the rows of Q satisfies all the requirements like 4.4.14 and then by choice of remaining rows of Q as in 4.4.13 satisfies all relations like 4.4.12, qualifying Q as the required transformation matrix.

The last element of the first column of 4.4.14 is $q(1)_{1*}\,\bar{A}^{(d_1-1)}\bar{B}_{*1}$ and it is equal to one as $q(1)_{1*}$ is the d_1th row of L^{-1}, while $\bar{A}^{(d_1-1)}\,\bar{B}_{*1}$ is the d_1th column of L. All

the other elements of first column of 4.4.14 are euqal to zero as, they are the products of d_1 th row of L^{-1} with j^{th} columns of L for $j = 1,2, \dots d_1-1$.

The second column of 4.4.14 has elements $q(1)_{1*} \bar{B}_{*2}$, $q(1)_{1*} \bar{A} \bar{B}_{*2} \dots q(1)_{1*} \bar{A}^{d_1-1} \bar{B}_{*2}$. All these will be zero if $d_1 \leq d_2$, because in that case $\bar{B}_{*2} \bar{A} \bar{B}_{*2} \dots \bar{A}^{d_1-1} \bar{B}_{*2}$ are columns of L other than the d_1 th i.e. σ_1 th column and $q(1)_{1*}$ is the d_1 th column of L^{-1}.

If $d_1 > d_2$, $\bar{A}^{d_1-1} \bar{B}_{*2}$ will not be a column of L. But it can be expressed as a linear combination of all the columns of $[\bar{B}, \bar{A} \bar{B}, \dots \bar{A}^{d_1-2} \bar{B}]$ and the column $\bar{A}^{d_1-1} \bar{B}_{*1}$. Thus $q(1)_{1*} \bar{A}^{d_1-1} \bar{B}_{*2}$ can be expressed as the linear combination of all the elements of $[q(1)_{1*} \bar{B}, \dots q(1)_{1*} \bar{A}^{d_1-2} \bar{B}]$ and the element $q(1)_{1*} \bar{A}^{d_1-1} \bar{B}_{*1}$ the only nonzero element of these being the last one equal to one. Thus the last element of second column may or may not be equal to zero, if $d_1 > d_2$. All the previous elements of second column of 4.4.14 are linear combinations of some of the elements of $[q(1)_{1*} \bar{B}, q(1)_{1*} \bar{A} \bar{B} \dots q(1)_{1*} \bar{A}^{d_1-2} \bar{B}]$ all of which are equal to zero. Similarly we can show the remaining columns of 4.4.14 have the required form.

Finally by writing equations similar to 4.4.11 for remaining rows of 4.4.10 it can be shown that the required transformation matrix is

$$Q = \begin{bmatrix} q_1 \\ q_1 \bar{A} \\ q_1 \bar{A}^2 \\ \vdots \\ q_1 \bar{A}^{d_1-1} \\ q_2 \\ q_2\bar{A} \\ \vdots \\ q_2 \bar{A}^{d_2-1} \\ \vdots \\ q_m \bar{A}^{d_m-1} \end{bmatrix}$$

with $q_i = q^{(i)}{}_{1*} = \sigma_i{}^{th}$ row of L^{-1} ...(4.4.17)

Luenberger(L1) has given the proof of the fact that Q is nonsingular.

If we rearrange the columns of given $\bar{B}$, which is equivalent to rearranging the input vector, we can satisfy the condition

$d_1 \leq d_2 \leq d_3 \ldots \leq d_m$ to get B in the form

$$B = \begin{array}{c} \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \sigma_1{}^{th} \text{ row} \\ \\ \\ \\ \sigma_2{}^{th} \text{ row} \\ \\ \\ \sigma_m{}^{th} \text{ row} \end{array} \quad \ldots(4.4.18)$$

B of 4.4.7 and 4.4.8 can also be modified to the form 4.4.18 by premultiplication by an elementary matrix which is realised by constant coefficient precompensator in the forward path.

The controllable companion form of B described above, with all except the σ_kth rows zero is specially suitable for linear state variable feedback compensation. The compensated system has its evolution matrix A+BF. It is easy to varify that with the particular form of B, the form of A+BF happens to be the same as of A.

We can modify the method of selecting the n linearly independent columns of M_c forming L, retaining the forms of A and B useful for linear state variable feedback design. For that, let L be written in the form

$$L = \left[\bar{B}_{*1}\ \bar{A}\bar{B}_{*1} \dots \bar{A}^{d_1-1}\bar{B}_{*1}\ \bar{B}_{*2}\ \bar{A}\bar{B}_{*2} \dots \bar{A}^{d_r-1}\bar{B}_{*r}\right] \qquad \dots(4.4.19)$$

where the choice of d_i's is made under the following constraints but otherwise freely,

i) $\sum_{i=1}^{r} d_i = n$ with $r \leq m$

ii) L is non-singular, and

iii) $|d_i - d_j| \leq 1$
for i, j = 1,2,..., r $\qquad \dots(4.4.20)$

If such a choice of L is possible, the transformation matrix Q similar to 4.4.17 will give the same form of A described by 4.4.7 and 4.4.8, with m replaced by r. but with the form of B as given

$$
B = \begin{bmatrix}
0 & 0 & \cdot & \cdot & \cdot & 0 & x & \cdot & \cdot & \cdot & x \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & 0 & \cdot & & & & \cdot \\
x & x & \cdot & \cdot & \cdot & x & \cdot & & & & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
x & \cdot & \cdot & \cdot & \cdot & x & \cdot & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
x & \cdot & \cdot & \cdot & \cdot & x & x & \cdot & \cdot & \cdot & x
\end{bmatrix} \qquad \ldots(4.4.21)
$$

Note that in 4.4.21, if the first r columns are considered, all except the $\sigma_k{}^{th}$ rows are zero. The remaining m-r columns do not have any specific form. With such a B it is possible to keep the form A+BF similar to 4.4.7 and 4.4.8 if F has its last m-r rows zero. These forms are useful to design a system which is controllable from smaller than m number of inputs.

The controllable companion form of A is suitable, for writing its characteristic polynomial directly by inspection, when its off diagonal blocks can be made zero, as a special case. This can be done by linear state variable feedback i.e. by proper selection of F such that A+BF is in the form,

$$
\begin{bmatrix}
A_{11} & 0 & \cdot & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & & \cdot & \cdot & \cdot & \cdot \\
\cdot & & & \cdot & \cdot & 0 \\
0 & \cdot & \cdot & \cdot & 0 & A_{rr}
\end{bmatrix} \qquad \ldots(4.4.22)
$$

with $r \leq m$.

The characteristic polynomial for 4.4.22 is

$$\prod_{i=1}^{r} | sI - A_{ii} |$$

and each A_{ii} is in the scalar companion form.

Example 4.1

Let the given system have

$$\bar{A} = \begin{bmatrix} 0 & 0 & -3 \\ 2 & 0 & -7 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \bar{B} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

The controllability matrix

$$M_c = \begin{bmatrix} 1 & 1 & 0 & -3 & 0 & -3 \\ 0 & -1 & 2 & -5 & 0 & -13 \\ 0 & 1 & 0 & 1 & -2 & 5 \end{bmatrix}$$

$$\rho \left[M_c \right] = 3$$

Thus the given pair $(\bar{A}, \bar{B})$ is controllable and can be converted into controllable companion form. We can obtain the transformed pair (A, B) in the following different ways.

i) Obtaining the linearly independent columns and rearranging them to form, L of 4.4.15 gives

$$L = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

with $d_1 = 2$ and $d_2 = 1$

i.e. $\sigma_1 = 2$ and $\sigma_2 = 3$

Then

$$Q = \begin{bmatrix} q_1 \\ q_1 \, A \\ q_2 \end{bmatrix}$$

where q_1 and q_2 are the second and third rows of L^{-1}.

$$L^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad Q = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -3.5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A = Q \, \overline{A} \, Q^{-1}$$

$$= \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -3.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -3 \\ 2 & 0 & -7 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc:c} 0 & 1 & 0 \\ 6 & -1 & -6 \\ \hdashline -2 & 0 & 1 \end{array}\right]$$

and

$$B = Q \, \overline{B} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & -0.5 & -3.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$$

(A,B) are in the controllable companion form given by 4.4.7 and 4.4.8 with 2nd and 3rd rows significant as expected.

2) Under the constraints 4.4.20 we can choose, for $2 = m = 2$,

$$L = [\bar{B}_{*1} \; \bar{B}_{*2} \; \bar{A} \; B_{*2}] \quad \text{with } d_1 = 1 \text{ and } d_2 = 2 \quad \text{i.e. } \sigma_1 = 1,\ \sigma_2 = 3$$

$$= \begin{bmatrix} 1 & 1 & -3 \\ 0 & -1 & -5 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|L| \neq 0$$

$$\sigma_1 = d_1 = 1,\ \sigma_2 = d_1 + d_2 = 3,\ |d_1 - d_2| \not> 1$$

Then

$$L^{-1} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0.25 & 1.25 \\ 0 & -0.25 & -0.25 \end{bmatrix}$$

$$\therefore \quad Q = \begin{bmatrix} q_1 \\ q_2 \\ q_2 A \end{bmatrix} \quad \text{where} \quad \begin{array}{l} q_1 = \text{First row of } L^{-1} \\ q_2 = \text{Third row of } L^{-1} \end{array}$$

$$= \begin{bmatrix} 1 & -1 & -2 \\ 0 & -0.25 & -0.25 \\ -0.5 & 0.25 & 1.75 \end{bmatrix}$$

$$\therefore \quad A = Q \bar{A} Q^{-1}$$

$$= \begin{bmatrix} 1 & -1 & -2 \\ 0 & -0.25 & -0.25 \\ -0.5 & 0.25 & 1.75 \end{bmatrix} \begin{bmatrix} 0 & 0 & -3 \\ 2 & 0 & -7 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1.5 & -5 & 1 \\ -0.5 & -3 & -1 \\ 0.5 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -0 & 0 & 0 \\ 0 & 0 & 1 \\ 1.5 & 3 & 2 \end{bmatrix}$$

and $B = Q \bar{B}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -0.5 & 1 \end{bmatrix}$$

Note that the form of B is as given by 4.4.21 with $r = m$ and not as given by 4.4.8.

3) The third choice, again under the constraints 4.4.20, but with $r < m$ is illustrated below. Here

$$L = \left[\bar{B}_{*1}\ \bar{A}\,\bar{B}_{*1}\ \bar{A}^2\,\bar{B}_{*1}\right]$$

so that $d_1 = d_r = 3$ i.e. $r = 1$

Then

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and $|L| \neq 0$

$$L^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$Q = \begin{bmatrix} q_1 \\ q_1\ A \\ q_1\ A^2 \end{bmatrix}$$

where q_1 = third row of L^{-1}

$$\therefore \quad Q = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & -3.5 \end{bmatrix}$$

$$\therefore \quad A = Q\,\bar{A}\,Q^{-1}$$

$$= \begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 0.5 & 0 \\ 1 & 0 & -3.5 \end{bmatrix} \begin{bmatrix} 0 & 0 & -3 \\ 2 & 0 & -7 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -7 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 7 & 0 \end{bmatrix}$$

$$B = Q\,\bar{B} \qquad \begin{bmatrix} 0 & -0.5 \\ 0 & -0.5 \\ 1 & -2.5 \end{bmatrix}$$

which is in the form given by 4.4.21 with $r = 1$.

4) Multivariable Uncontrollable Companion Form

The procedure of transforming the given controllable pair $(\bar{A}, \bar{B})$ into multivariable controllable companion form requires n columns of controllability matrix M_c to be linearly independent. If the given pair $(\bar{A}, \bar{B})$ is not controllable, the maximum possible number of linearly independent columns of M_c is $\bar{n} < n$. In this case, the following procedure can be followed to transform $(\bar{A}, \bar{B})$ in to the canonical forms partly similar to the forms 4.4.7 and 4.4.8 and in addition, separating the controllable and uncontrollable parts of A. The resulting form of (A, B) can be called the uncontrollable companion form.

The $\bar{n}$ linearly independent columns of M_c selected from left to right to form L of 4.4.15 are amended by $n - \bar{n}$ arbitrary

columns such that the resulting set of n columns is linearly independent set. This L is then used to form the transformation matrix Q as per the procedure described for multivariable controllable companion form.

The transformed pair (A, B) obtained using Q has the form shown below (W-1) :

$$A = \left[\begin{array}{c|c} A_c & A_n \\ \hline 0 & A_{uc} \end{array}\right] \qquad \ldots(4.4.23)$$

where A_c is $\bar{n} \times \bar{n}$ matrix in the mv controllable companion form with the controllability indices d_i such that

$$\sum_{i=1}^{m} d_i = \bar{n} \qquad \ldots(4.4.24)$$

Note that because of 4.4.24, the Q defined as in 4.4.17 will have only $\bar{n}$ rows. The last $n - \bar{n}$ rows of L^{-1}, for L defined above for the uncontrollable companion form.

The form of B corresponding to the form 4.4.23 of A is

$$B = \left[\begin{array}{c} B_c \\ \hline 0 \end{array}\right] \qquad \ldots(4.4.25)$$

where the $\bar{n} \times m$ matrix B_c is the controllable companion form and last $n - \bar{n}$ rows of B are zero. From the specific forms obtained for A, B it is clear that they get partitioned in such a way as to separate the controllable and uncontrollable parts. The characteristic polynomial of A of 4.4.23 is (e-1)

$$|sI_{\bar{n}} - A_c|\cdot|sI_{(n-\bar{n})} - A_{uc}| \qquad \ldots(4.4.26)$$

separating the controllable and uncontrollable poles of the system.

4.5 Determination of the Rank

The first m columns of the controllability matrix M_c which are also the columns of $\bar{B}$, can be assumed linearly independent. The remaining (nm - m) columns are to be tested for linear independence. For low order systems it can be decided just by inspection. But for higher order systems the $(m + 1)^{th}$ onwards columns can be tested for linear independence by one of the following systematic methods.

i) Reduction to Echelon Form : By simple column operations the given n x r matrix can be reduced to the standard Echelon form having following properties :

a) The first nonzero element in each column is 1.
b) In any row containing the first nonzero element of any column that element is the only nonzero element of that row.
c) The zero columns, if any, come last.
d) The leading 1's in the nonzero columns are along a broken line, sloping downwards.

The nonzero columns of this transformed matrix in Echelon form are linearly independent. This determines the rank of this matrix and hence of the original matrix M_c, being column equivalent.

Example 4.2 : Transform the following matrix to column reduced Echelon form.

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 7 & 1 & 4 \end{bmatrix}$$

Let us denote the column operation 'column two is replaced by column 2 plus column 1 multiplied by -3' by C_{21} (-3)

i.e. $C_{ij}(k)$ - means i^{th} column is replaced by (i^{th} column) + (j^{th} column multiplied by k).

Similarly the other notations for column operations are

C_{ij} - interchange i^{th} and j^{th} columns

$C_i(k)$ - multiply i^{th} column by k

Reduction of the given A :

1) $C_{21}(-3)$, $C_{41}(-1)$ $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -4 & 2 & 1 \\ 3 & -8 & 4 & -2 \\ 1 & -2 & 1 & 0 \\ 3 & -2 & 1 & 1 \end{bmatrix}$

2) C_{24} $C_2(-1)$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & -4 \\ 3 & 2 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 3 & -1 & 1 & -2 \end{bmatrix}$

3) $C_{12}(-2),\ C_{32}(-2),\ C_{42}(4) \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 5 & -1 & 3 & -6 \end{bmatrix}$

4) $C_{43}(2),\ C_{13}(-1) \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix}$

The linear independence of rows of any matrix can be tested in similar manner just by replacing the word column by row in the above description.

ii) Gram Matrix Criterion : The columns of M_c can be tested applying this criterion. The columns say $C_1, C_2, \ldots, C_r$ are linearly independent iff the matrix

$$G = \begin{bmatrix} C_1^T C_1 & C_2^T C_1 & \cdots & C_r^T C_1 \\ \vdots & \vdots & & \vdots \\ C_1^T C_r & \cdot & \cdots & C_r^T C_r \end{bmatrix} \qquad \ldots(4.5.1)$$

is nonsingular.

If the system order is very large computer has to be used to program the test of linear independence of the columns

of C matrix. The $m+1^{th}$ column is tested for l.i. by any of the above methods. Then the test is repeated for the n x (m+2) matrix including $(m+2)^{th}$ column end so on. If any column A^kB_{*1} is found to be linearly dependent, all A^rB_{*i} for $r \geq k$ are linearly dependent and need not be subjected to test. This can be proved as follows.

If A^kB_{*i} is linearly dependent on previous columns we can write for some nonzero column vector α,

$$B_{*1}\ \alpha_1 + AB_{*2}\ \alpha_2 + \ldots A^2B_{*1}\ \alpha_{m+1} \ldots + A^{k+1}B_{*i}\ \alpha_{km+i} = 0 \qquad \ldots(4.5.3)$$

Adding 4.5.2 and 4.5.3

$$B_{*1}\ \alpha_1 + \ldots B_{*m}\alpha_m + AB_{*1}(\alpha_1+\alpha_{m+1}) \ldots + A^{k+1}B_{*i}\ \alpha_{km+i} = 0$$

Thus $A^{k+1}\ B_{*i}$ is linearly dependent.

The next column to be tested is $A^kB_{*(i+1)}$ and so on.

The rank of can be determined by finding out the max. no. of linearly independent columns by the above procedure.

4.6 Frequency Domain Interpretation of Controllability

The frequency domain description of a system may be obtained by Laplace transformation from any of the time domain representations, either state space or differential operator.

i) Let us consider the frequency domain description obtained from the S.S. description,

$$\dot{x} = Ax + Bu$$
$$\dot{y} = Cx + Eu \qquad \ldots(4.6.1)$$

By Laplace transformation

$$x(s) = (sI-A)^{-1} Bu(s)$$

$$= \frac{(sI-A)^{+} B}{|sI-A|} u(s) \qquad \ldots(4.6.2)$$

The system controllability can be found in the frequency domain as follows. Calculate

$$[sI-A]^{-1} = \frac{[sI-A]^{+}}{|sI-A|} = \frac{N(s)}{p(s)} \qquad \ldots(4.6.3)$$

where $\frac{N(s)}{p(s)}$ is in the simplified form after all possible cancellations. Then the system is controllable if there are no cancellations in

$$\frac{N(s)\,B}{p(s)} \qquad \ldots(4.6.4)$$

If there are any cancellations, the system is uncontrollable. It can be converted to the uncontrollable companion form of (A, B) to get the uncontrollable states. The cancellable factors of $|sI-A|$ give the modes of these uncontrollable states, as explained later in section 7.3. For the time being, the same can be illustrated by a simple example.

Example 4.3 : Let $A = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix}$

We can show, using the frequency domain criterion, that if $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the system is controllable. But for $B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ the system is uncontrollable.

Let $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$[sI-A]^{-1} = \begin{bmatrix} s+3 & -1 \\ 2 & s-1.5 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s-1.5 & 1 \\ -2 & s+3 \end{bmatrix}}{(s+2.5)\,(s-1)}$$

$$\therefore \quad [sI-A]^{-1} B = \frac{1}{(s+2.5)(s-1)} \begin{bmatrix} 1 \\ s+3 \end{bmatrix}$$

There is no cancellation and hence the system is controllable. This can be verified by the fact that its M_c matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 1.5 \end{bmatrix}$ with rank 2.

Now let $B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Then,

$$[sI-A]^{-1} B = \begin{bmatrix} s + 2.5 \\ 4\ (s+2.5) \end{bmatrix} \frac{1}{(s+2.5)(s-1)}$$

Thus the factor (s+2.5) of the characteristic polynomial cancels from each element of $(sI-A)^{-1}$ B. The system can be converted into equivalent uncontrollable companion form using the algorithm explained already. Now,

$$M_c = \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \quad \therefore \rho[M_c] = 1$$

Let $\quad \bar{L} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ with second column chosen arbitrarily to make $[\bar{L}] = 2$

Then the transformation matrix Q_e is

$$Q_e = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \text{First row of } L^{-1} \\ \text{Second row of } L^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}.$$

$$\therefore \quad \bar{B} = Q\,B = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ in the

expected form, showing $\bar{x}_2$ to be uncontrollable

$$\bar{A} = Q_e\ A\ Q_e^{-1} = \left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & -2.5 \end{array}\right]$$

showing that the root of the uncontrollable part is 2.5 and confirming that it is corresponding to the cancelled factor in $(sI-A)^{-1} B$.

ii) The other frequency domain description can be obtained from D.O. representation as

$$P(s)\ z(s) = Q(s)\ u(s) \quad \ldots(4.6.5)$$

i.e. $z(s) = P^{-1}(s)\ Q(s)\ u(s)$

Let
$$P(s) = G_L(s)\ \overline{P}(s)$$
$$Q(s) = G_L(s)\ \overline{Q}(s) \quad \ldots(4.6.6)$$

Then $G_L(s)$ is called the common left divisor of $P(s)$ and $Q(s)$. If $G_L(s)$ is such that $|G_L(s)|$ is of largest possible degree, then $G_L(s)$ is called the greatest common left divisor (g.c.l.d.) of $P(s)$ and $Q(s)$.

The system represented by 4.6.3 is controllable if and only if the g.c.l.d. of $P(s)$ and $Q(s)$ is unimodular. Such $P(s)$ and $Q(s)$ are said to be relatively left prime.

The method of finding out g.c.l.d. of given $P(s)$ and $Q(s)$ is explained in the next chapter along with its counterpart in the study of observability.

The proof of the above mentioned frequency domain criterian of controllability will be obvious after studying the minimal systems in chapter -7.

<u>Exercise 4</u> :-

(4.1) Consider the following control systems having

(i) $$e^{At} = \begin{bmatrix} e^{t} & & \\ & e^{-t} & \\ & & e^{2t} \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(ii) $$e^{At} = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} & 0 \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Are the rows of e^{At} B linearly independent over the real field?

(b) What is the rank of e^{At} B in each case?

(c) From (i) comment on the controllability of each system. Verify the result by calculation of controllability matrix in each case.

(4.2) Are the following (A, B) pairs controllable?

(a) $$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} -2 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -3 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

(4.3) 1. Trasform the pair (A, B) of 4.2(b) into the forms :

i) given by 4.4.7 and 4.4.8

ii) given by 4.4.22 and 4.4.21 with

a) $d_1 = 1,\ d_2 = 1,\ d_3 = 2$

b) $d_1 = d_2 = 2$

c) $d_1 = 4$

d) $d_1 = 1,\ d_2 = 3$

e) $d_1 = 3,\ d_2 = 1$

2. From how many inputs each of the six systems can be controlled?

3. Which of the six forms of transformed (A, B) are suitable for linear state variable feedback?

(4.4) Determine the controllable and uncontrollable poles of the control system having

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Determine the transfer matrix T(s), if

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \quad D = 0$$

Consider only the controllable parts of the transformed matrices $\bar{A}$ and $\bar{B}$. Obtain corresponding $\bar{C}$ and the transfer matrix. How do the two transfer matrices compare?

(4.5) The M_c matrix for the pair (A, B) of problem 4.3 is

$$M_c = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine the rank of M_c using column reduction to Echelon form.

(4.6) Give proof for the fact that the controllable companion form of B is as in (4.4.18) if $d_1 \leq d_2 \leq d_3 \dots \leq d_m$.

(4.7) Convert the following matrix to row reduced Echelon form:

$$\begin{bmatrix} 15 & 35 & 18 & 12 & -31 & 12 \\ -2 & -8 & -5 & -5 & 9 & -3 \\ 12 & 23 & 13 & 4 & -21 & 9 \\ 7 & 21 & 11 & 10 & -19 & 7 \\ 11 & 20 & 10 & 4 & -17 & 7 \\ -7 & 19 & 10 & 4 & 14 & -7 \end{bmatrix}$$

(4.8) Show that the characteristic polynomial for the matrix A of the form 4.4.23 is given by 4.4.26.

(4.9) Prove that the uncontrollable companion form of the pair (A, B), given by (4.4.23) and (4.4.25), is obtained by the given procedure.

+ + +

CHAPTER - 5

OBSERVABILITY

5.1 INTRODUCTION

It is necessary to know the dynamic system state either for analysis or for feedback. But all the state variables of a practical system are not always available for measurement or feedback. Thus it is desirable that it should be possible to compute the system state from the measurable signals which are the inputs and the outputs of the system. This property is called observability.

5.2 DEFINITION AND CRITERIA

A system described by the state equations

$$\dot{x} = Ax + Bu$$

$$y = Cx + Eu \qquad \ldots(5.2.1)$$

is observable, if for given y and u over a time interval t_o to t_1, x can be determined over that interval, which in turn requires determination of $x(t_o)$. Let,

$$\bar{y} = y - Eu \qquad \ldots(5.2.2)$$

Thus for given y and u, $\bar{y}$ is known and from 5.2.1

$$\bar{y} = Cx \qquad \ldots(5.2.3)$$

Thus the system is observable if 5.2.3 can be solved for x. Substituting for x from 3.4.2,

$$\bar{y} - C \int_{t_o}^{t} e^{A(t-\tau)} B.u(\tau)\, d\tau$$

$$= C\, e^{A(t-t_o)}\, x(t_o) \qquad \ldots(5.2.4)$$

Assuming $t_o = o$, $\tilde{y}$ the l.h.s. of 5.2.4 which is a function of known $\bar{y}$ and u can be written as

$$\tilde{y} = C\, e^{At} x_o \qquad \ldots(5.2.5)$$

where $x_o = x(o)$

Substituting for e^{At} from 4.3.4

$$\tilde{y} = C\,[K_o(t)I_n + K_1(t)A + K_2(t)A^2 \ldots + K_{n-1}(t)A^{n-1}]\,x_o$$

$$= [K_o(t)CI_n + K_1(t)CA + \ldots K_{n-1}(t)CA^{n-1}]\,x_o$$

$$= [K_o(t)I_p,\ K_1(t)I_p,\ \ldots K_{n-1}(t)I_p] \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} x_o \qquad \ldots(5.2.6)$$

In the above equation x_o, i.e. n components of the vector x_o, are to be determined. At any known time t_1, 5.2.6 gives p equations. We can add required number of equations to these p equation by writing 5.2.6 for $t = t_2$, t_3 etc to get total n equations of the form

$$y_n = M \cdot \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} x_o \qquad \ldots(5.2.7)$$

The initial state x_o can be calculated by inverting M and the observability matrix

$$M_o = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} \qquad \ldots(5.2.8)$$

As M is nonsingular, the necessary and sufficient condition of observability is that the rank of M_o is n.

Other criteria for observability are listed below.

1) $V(t_o,t_1) = \int_{t_o}^{t_1} e^{A^T\tau} C^T C \, e^{A\tau} d\tau$

is nonsingular.

2) The n columns of Ce^{At} are linearly independent for all $t(o,\infty)$ over the real field R.

3) No cancellation in $C(sI-A)^{-1}$, other than those in $(sI-A)^{-1}$ itself.

4) If the D.O. representation of the system is

$$P(s)\ z(s) = u(s)$$

and $$y(s) = R(s)\ z(s),$$

then the system is observable if and only if the g.c.r.d. of R(s) and P(s) is unimodular. Such R(s) and P(s) are said to be relatively right prime.

The proofs are similar to those of controllability criteria. The duality observed in these two characteristics, controllability and observability is explained in the next section.

5.3 DUALITY

By observing the controllability matrix

$$M_c = [B \; AB \; A^2B \; \dots \; A^{n-1}B] \qquad \dots(5.3.1)$$

and the observability matrix

$$M_o = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} \qquad \dots(5.3.2)$$

it is easy to note that if a system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Eu \end{aligned} \qquad \dots(5.3.3)$$

is controllable and/or observable then the system

$$\begin{aligned} \dot{\bar{x}} &= A^T \bar{x} + C^T \bar{u} \\ \bar{y} &= B^T \bar{x} + E^T \bar{u} \end{aligned} \qquad \dots(5.3.4)$$

is observable and/or controllable respectively and vice versa. Hence the systems 5.3.3 and 5.3.4 are called dual systems. The observable companion forms (A_o, C_o) of an observable pair (A,C) are given below. Note that they can be obtained by transposing the forms given in 4.4.7 and 4.4.8.

$$A_o = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ A_{P1} & \cdot & \cdots & A_{pp} \end{bmatrix} \qquad \dots(5.3.5)$$

where the diagonal block A_{jj} are in the scalar observable companion form,

$$A_{jj} = \begin{bmatrix} 0 & 0 & . & . & 0 & -a_0 \\ 1 & 0 & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 0 & . \\ 0 & 0 & . & . & 1 & -a_{n-1} \end{bmatrix} \qquad \ldots(5.3.6)$$

A_{jj} is of dimension $d_j \times d_j$ where d_j for $j = 1,2 \ldots p$ are the observability indices. The maximum of d_j is called observability index of the system and is denoted by ν.

$$\sum_{j=1}^{p} d_j = n \qquad \ldots(5.3.7)$$

The off diagonal blocks A_{ij} have all columns zero except the last one. Its dimension is $d_i \times d_j$ with $i \neq j$. Thus A_o has only p significant columns numbered

$$\sigma_k = \sum_{j=1}^{k} d_j \qquad \ldots(5.3.8)$$

for $k = 1, 2 \cdots p$.

The matrix C_o also has the same numbered p columns significant and all other columns zero, thus haging the form

$$C_o = \begin{bmatrix} 0 & . & 0 & 1 & 0 & . & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & 0 \\ 0 & . & 0 & x & 0 & . & 0 & x & . & . & . & 1 \end{bmatrix} \qquad \ldots(5.3.9)$$

The system matrices (A, C) can be converted to the forms 5.3.5 and 5.3.9 if the system is observable. The algorithm for

this equivalent transformation is similar to that for controllable companion forms and can be worked out using dual concepts. The unobservable system matrices (A,C) can be converted to the unobservable companion forms which are transposes of the forms 4.4.23 and 4.4.25. Algorithm for getting unobservable companion forms is similar to that for getting uncontrollable companion forms given by 4.4.23 and 4.4.25.

The proofs of the frequency domain criteria of observability can also be given easily by applying duality. Application of these frequency domain criteria requires determination of the g.c.l.d. and g.c.r.d. of two polynomial matrices which in turn require triangularisation of polynomial matrices. The computation methods for these are given in the following sections.

5.4 G.C.L.D. and G.C.R.D.

The greatest common left divisor (g.c.l.d.) of two p x p and p x m matrices P(s) and Q(s) can be derived as follows. Write the p x (p+m) composite matrix

$$[P(s) \,\vdots\, Q(s)] \qquad \ldots(5.4.1)$$

Convert it by simple column operations to a lower left triangular form as explained in the next section. This transformation can be represented by

$$[P(s) \,\vdots\, Q(s)]\, U_R(s) = \begin{bmatrix} X & 0 & . & . & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . & & & . \\ . & . & . & . & . & . & & & . \\ . & . & . & . & 0 & . & & & . \\ X & . & . & . & X & 0 & . & . & 0 \end{bmatrix} \qquad \ldots(5.4.2)$$

$$\therefore \quad [P(s) \;\vdots\; Q(s)] = [T_m(s) \;\vdots\; 0]\, U_R^{-1}$$

$$= [T_m(s) \;\vdots\; 0] \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \qquad \ldots(5.4.3)$$

$$\therefore \quad P(s) = T_m(s)\, U_1(s)$$

$$Q(s) = T_m(s)\, U_2(s) \qquad \ldots(5.4.4)$$

Thus $T_m(s)$ is a common left divisor of $P(s)$ and $Q(s)$. To show that $T_m(s)$ is a g.c.l.d., write 5.4.2 with $U_R(s)$ in the partioned form as,

$$[P(s) \;\vdots\; Q(s)] \begin{bmatrix} U_5 & U_6 \\ U_7 & U_8 \end{bmatrix} = [T_m(s) \;\vdots\; 0] \qquad \ldots(5.4.5)$$

$$P(s).U_5(s) + Q(s).U_7(s) = T_m(s) \qquad \ldots(5.4.6)$$

If $G_L(s)$ is any common left divisor of $P(s)$ and $Q(s)$, 5.46 can be written as

$$G_L(s)\, [\bar{P}(s)\, U_5(s) + \bar{Q}(s)\, U_7(s)] = T_m(s) \qquad \ldots(5.4.7)$$

Thus any common left divisor of $P(s)$ and $Q(s)$ is also a left divisor of $T_m(s)$. Hence $T_m(s)$ is g.c.l.d. of $P(s)$ and $Q(s)$. Note that a g.c.l.d. is not unique because any column operations on $T_m(s)$ will give rise to another g.c.l.d. The order of the determinant of the g.c.l.d. is unique for given polynomial matrices $P(s)$ and $Q(s)$. This order is maximum compared to that of the determinant of any other common left divisor of $P(s)$ and $Q(s)$.

Similar method can be used to find the greatest common right divisor (g.c.r.d.) of two matrices R(s) and P(s) of dimensions p x m and m x m respectively. In this case it is necessary to convert the composite matrix

$$\left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right] \qquad \ldots(5.4.8)$$

into an upper right triangular form by row operations, which can be written as,

$$U_L(s) \left[\begin{array}{c} R(s) \\ \hline P(s) \end{array}\right] = \begin{bmatrix} X & . & . & X \\ . & . & . & \\ . & . & . & . \\ 0 & . & . & X \\ 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \end{bmatrix} = \left[\begin{array}{c} T_m(s) \\ \hline 0 \end{array}\right] \qquad \ldots(5.4.9)$$

$T_m(s)$ is the g.c.r.d.

5.5 TRIANGULARISATION OF MATRICES

The column and row operations defined in Section 2.5 are used for triangularisation. The following procedure is used for converting a polynomial matrix to a lower left triangular form.

The first element of first row is made the one of lowest order in the first row by interchange of columns. Then by other column operations all but the first element of first row are made zero. Then the second row is considered. The part of the second row from its second element to the last element is treated in the same way as the first row. That is, the lowest

order element of this part is braught in the second place of second row and all other elements, 3^{rd} onwards, are made zero by column operation. All the rows are treated in succession in the same manner taking into account elements, i^{th} to the last one, of i^{th} row. The procedure is illustrated by example 5.1.

Similar procedure with row operations can be used for getting upper right triangular form.

Example 5.1 :

Show that the system described by

$$P(D)\ Z(t) = Q(D)\ u(t)$$

with

$$P(D) = \begin{bmatrix} D^2 & -1 \\ 0 & D^2 \end{bmatrix} \text{ and } Q(D) = \begin{bmatrix} 1 & D \\ D & 1 \end{bmatrix}$$

is controllable.

The composite matrix,

$$V(s) = [P(s) \vdots Q(s)]$$

$$= \begin{bmatrix} s^2 & -1 & 1 & s \\ 0 & s^2 & s & 1 \end{bmatrix}$$

Following the notation for column operations given in illustrative example 4.2, V(s) can be reduced to lower left triangular form as

$$\begin{bmatrix} s^2 & -1 & 1 & s \\ 0 & s^2 & s & 1 \end{bmatrix} \xrightarrow{C_{13}} \begin{bmatrix} 1 & -1 & s^2 & s \\ s & s^2 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{C_{21}(1) \\ C_{31}(-s^2) \\ C_{41}(-s)}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ s & s^2+s & -s^3 & 1-s^2 \end{bmatrix} \xrightarrow{\substack{C_{42}(1) \\ C_{32}(s)}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & s^2+s & s^2 & 1+3 \end{bmatrix} \xrightarrow{\substack{C_{24} \\ C_{42}(-s)}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1+s & s^2 & 0 \end{bmatrix} \xrightarrow{C_{32}(-s)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1+s & -s & 0 \end{bmatrix} \xrightarrow{C_{32}(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1+s & 1 & 0 \end{bmatrix} \xrightarrow[C_{32}(-1-s)]{C_{23}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad T_m(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \qquad |T_m(s)| = 1$$

$T_m(s)$, the g.c.l.d. of $P(s)$ and $Q(s)$ is unimodular. So the system is controllable.

5.6 EFFECT OF EQUIVALENT TRANSFORMATION

The properties controllability and observability of a system are not affected by equivalent transformation as shown below.

If a pair (A, B) is controllable, then

$$\rho[M_c] = \rho[B \quad AB \; \ldots . \; A^{n-1}B] = n \qquad \ldots(5.6.1)$$

Let

$$\bar{A} = QAQ^{-1} \qquad \ldots(5.6.2)$$

and $\bar{B} = QB$

with a nonsingular matrix Q.

The controllability matrix $\overline{M}_c$ for $(\overline{A}, \overline{B})$ is

$$\overline{M}_c = [QB, QAQ^{-1}QB \; \ldots \;]$$

$$= Q[B \; AB \; \ldots \; A^{n-1}B]$$

$$\therefore \quad \rho[\overline{M}_c] = n \qquad \ldots(5.6.3)$$

$\therefore$ The pair $(\overline{A} \; \overline{B})$ is also controllable.

Similarly it can be shown that observability is not affected by linear transformation.

Exercise 5 :

5.1 : Write the dual system quadruple when the given system quadruple is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Is the dual system observable ?

Is the given system controllable ?

5.2 : Is the following (A, C) pair observable? What is the minimum number of output variables from which its state can be observed?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 1 & 2 & 2 \\ 0 & 0 & -2 & 1 \\ 2 & 2 & -2 & 1 \end{bmatrix}$$

5.3 : Prove that for an observable system, the state can be observed by knowing the outputs and their (ν - 1) derivatives, where ν is the observability index.

5.4 : Convert the (A, C) of problem 5.2 to the equivalent multivariable observable companion form.

5.5 : Is the system, described by

$$T(s) = R(s)\, P^{-1}(s)$$

$$= \begin{bmatrix} S+1 & S \\ S+2 & S+2 \end{bmatrix} \begin{bmatrix} S^2 & 0 \\ 0 & S^2 \end{bmatrix}^{-1}$$

Observable? What is the g.c.r.d. of R(s) and P(s) ?

5.6 : For the system in the example 5.2, the B and E matrices are given by

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Is the system controllable?

How many times the output of the system has to be differentiated if we want to get the system state from the outputs and their derivatives?

CHAPTER - 6

REPRODUCIBILITY

6.1 POINTWISE OUTPUT CONTROLLABILITY

The concepts of output controllability and input observability are incorporated in the concept of reproducibility. The controllability discussed in Chapter-4 can specifically be called as state controllability. By consideration similar to that of Section 4.3 it can be proved that the criterion for output controllability is that the rank of the output controllability matrix

$$M_{oc} = \left[CB \; CAB \; \ldots \; CA^{n-1}B \right] \qquad \ldots(6.1.1)$$

should be p. The output controllability corresponding to this criterion can more specifically be called as pointwise output controllability.

The desired controllability of the output may be functional controllability. That is, it may be desired to produce an output function defined over an interval. The output function controllability then means the possibility of finding some u(t) to produce this output function in the given system.

Further, desired output may be reproduction of some output already known to exist in the system or it may be any output arbitrarily specified. To differentiate these two phases which are found to have slightly different requirements,

the first one is called input (function) observability and the later is called output (function) controllability where the function word in the parenthesis may be omitted after it is clearly understood. The two properties combinedly can be represented by the word reproducibility.

6.2 OUTPUT FUNCTION CONTROLLABILITY

The output of a system represented by the state space triple (A,B,C) is given by

$$y(s) = C(sI-A)^{-1} x_o + C(sI-A)^{-1} Bu(s) \qquad \ldots(6.2.1)$$

If x_o of the system is specified along with the desired y(s), a new $\bar{y}(s) = y(s) - C(sI-A)^{-1} x_o$ can be defined and made equal to $C(sI-A)^{-1} Bu(s)$. If x_o is not specified it can be assumed equal to zero. Thus in any case output controllability requires the possibility of finding out u(s) for arbitrarily specified y(s), that is,with its p elements forming a linearly independent set and chosen from the set of possible outputs.

$$y(s) = T(s)\, u(s) \qquad \ldots(6.2.2)$$

where $T(s) = C(sI-A)^{-1} B \qquad \ldots(6.2.3)$

The dimension of the output vector is p. An arbitrary specification of p components of y(s) requires that none of them should be linearly dependent on others. This further requires that none of the rows of T(s) should be linearly dependent on the others. Hence it is necessary that

$$\rho\,[T(s)] = p \qquad \ldots(6.2.4)$$

which obviously requires $p \leq m$. Thus the system cannot be output function controllable if the number of inputs is less than the number of outputs.

$\rho[T(s)] = p$, means that the p rows of the transfer matrix are linearly independent over the field $\mathcal{P}$ of rational functions i.e. the smallest field which contains the elements of T(s). The property of output controllability further needs the possibility of determining an input to satisfy equation 6.2.2.

If we can define an m x p matrix $T_{RI}(s)$ such that

$$\underset{p \times m}{T(s)} \; \underset{m \times p}{T_{RI}(s)} = I_p \qquad \ldots(6.2.5)$$

then we can find a u(s) given by $T_{RI}(s)$ y(s) which substituted in 6.2.2 satisfies the equation. $T_{RI}(s)$ satisfying 6.2.5 is called as a right inverse of T(s). It may be noted that rank of I_p is p. Hence the necessary condition for the existence of the right inverse $T_{RI}(s)$ is that the rank of T(s) be p. It can be shown constructively that this is also the sufficient condition for the existence of $T_{RI}(s)$. One construction for finding $T_{RI}(s)$ is given later in Chapter-11, refer equation (11.2.17). The matrix equation 6.2.5 is equivalent to p^2 scalar equations equating the corresponding elements on the l.h.s. and r.h.s. of 6.2.5. The no. of elements of $T_{RI}(s)$ to be chosen arbitrarily is pm which is larger than p^2, as m > p. Thus the arbitrary choice of $T_{RI}(s)$ cannot be unique. However existence of $T_{RI}(s)$ is sufficient for obtaining u(s) giving the arbitrarily specified desired output. Hence (6.2.4) is the sufficient as well as necessary condition for output function controllability. The input u(s) satisfying (6.2.2) can be written as

$$\mathbf{u}(s) = T_{RI}(s)\ y_d(s) \qquad \ldots(6.2.6)$$

for given $y(s) = y_d(s)$ ...(6.2.7)

Hence $\mathbf{u}(s)$ given by 6.2.6 is also not unique. Note that by definition the system is output function controllable if a $u(t)$ can be found to produce the desired arbitrarily specified output.

Frequency domain is chosen for analysis of output controllability and determination of existence of a (nonunique) control. This is simpler than finding conditions of output function controllability in time domain.

Brockett and Measarovic [B.1] have derived the condition output controllability in time domain. It states that a system described by a triple (A, B, C) is output function controllable if and only if the (np)x(2 nm) matrix

$$m_p = \begin{bmatrix} CB & CAB & \ldots & CA^{n-1}B & \ldots & CA^{2n-1}B \\ 0 & CB & \ldots & CA^{n-2}B & \ldots & CA^{2n-2}B \\ 0 & 0 & \ldots & & & \\ 0 & \cdot & \ldots & CB & & CA^{n}B \end{bmatrix}$$

is of rank np. ...(6.2.8)

Note that this requires that all the np rows of m_p should be linearly independent. This satisfies automatically the

condition for pointwise controllability. Thus any output function controllable system must be pointwise output controllable. It is obvious that criterion 6.2.4 with $T(s)$ in 6.2.3 is much easier to apply than 6.2.8.

We can study the example given by Birta and Mufti (B-2) to illustrate clearly the above concepts.

<u>Example 6.1</u> : Let a system be described by a state space triple (A, B, C) given as :

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad p = m = 2,\ n = 3 \qquad \ldots(6.2.9)$$

This system satisfies the conditions for controllability observability and pointwise output controllability. But still it is not output function controllable because

$$T(s)=C(sI-A)^{-1}B$$

$$= \begin{bmatrix} -s & s^2 \\ 1 & -s \end{bmatrix} \qquad \ldots(6.2.10)$$

for the (A, B, C) of 6.2.9

$$\therefore \quad |T(s)| = 0 \qquad \ldots(6.2.11)$$

i.e. $\rho\,|T(s)| \neq 2$

Note that the rows of 6.2.10 are not linearly independent over ρ, the field of rational functions. The first row can be obtained by multiplying the second one by $-s$

Example 6.2 : Let the triple (A, B, C) describing a system be given by :

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \qquad \ldots(6.2.12)$$

It can be shown that the system is uncontrollable and unobservable, (as far as the complete state is concerned). But it can be shown that it is output function controllable :

$$\begin{aligned} T(s) &= C(sI-A)^{-1}B \\ &= \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & 0 \\ 0 & 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{s+1} \end{aligned} \qquad \ldots(6.2.13)$$

$\therefore$ T(s) has rank = 1

and for any y(s), the corresponding u(s) is given by (s+1) y(s). Note that the input can be determined uniquely in this case. This is true whenever T(s) is a square matrix having rank p = m.

By applying the criterion for pointwise output controllability it can be verified that the system is pointwise output controllable as expected.

Example 6.3 : Let the system be described by the triple :

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \qquad \ldots(6.2.14)$$

$$T(s) = C(sI-A)^{-1} B$$

$$= \begin{bmatrix} \dfrac{3s+4}{(s+1)\ (s+2)} & \dfrac{1}{s+1} \end{bmatrix} \qquad \ldots(6.2.15)$$

There is a single output. Hence it can be specified arbitrarily. So the system is output controllable [or in other words rank of the 1 x 2 matrix T(s) is 1].

Let the specified desired y(t) be :

$$y(t) = 2 - e^{-t} - e^{-2t} - te^{-t} \qquad \ldots(6.2.16)$$

Findout a u(t) which can give this y(t). Is this u(t) unique?

$$\therefore \quad y(s) = \frac{2}{s} - \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{(s+1)^2} \qquad \ldots(6.2.17)$$

$$= \frac{1}{s} \left[\frac{3s+4}{(s+1)(s+2)} \right] + \frac{1}{(s+1)^2}$$

Let $^1T_{RI}(s) = \begin{bmatrix} \dfrac{0.5(s+1)\ (s+2)}{3s+4} \\ 0.5(s+1) \end{bmatrix}$

be a possible right inverse of T(s)

so that $T(s)^1\ T_{RI}(s) = 1 \qquad \ldots(6.2.19)$

Let $U_1(s) = {}^1T_{RI}(s)\ y(s) \qquad \ldots(6.2.20)$

$$\therefore \quad U_1(s) = \begin{bmatrix} \dfrac{0.5(s+1)\ (s+2)}{3s+4} \\ 0.5(s+1) \end{bmatrix} \left[\frac{1}{s} \left\{ \frac{3s+4}{(s+1)(s+2)} \right\} + \frac{1}{(s+1)^2} \right]$$

$$= \begin{bmatrix} \dfrac{0.5}{5} + \dfrac{0.5(s+2)}{(1+s)\ (3s+4)} \\ \dfrac{0.5(3s+4)}{s(s+2)} + \dfrac{0.5}{1+s} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{0.5}{s} + \dfrac{0.5}{1+s} - \dfrac{1}{3s+4} \\ \dfrac{1}{s} + \dfrac{0.5}{s+2} + \dfrac{0.5}{1+s} \end{bmatrix} \qquad \ldots(6.2.2)$$

$$\begin{bmatrix} 0.5 + 0.5\ e^{-t} - \frac{1}{3}\ e^{-\frac{4}{3}t} \\ 1 + 0.5\ e^{-2t} + 0.5\ e^{-t} \end{bmatrix} \qquad \ldots(6.2.22)$$

It can be verified that

$y(s) = T(s)\ U_1(s)$

$$\begin{bmatrix} \dfrac{3s+4}{(s+1)\ (s+2)} & \dfrac{1}{s+1} \end{bmatrix} \begin{bmatrix} \dfrac{0.5}{s} + \dfrac{0.5}{1+s} - \dfrac{1}{3s+4} \\ \dfrac{1}{s} + \dfrac{0.5}{s+2} + \dfrac{0.5}{1+s} \end{bmatrix}$$

$$= \frac{1}{(s+1)^2} + \frac{3s+4}{s(s+1)(s+2)} \qquad \ldots(6.2.23)$$

Let another choice of $T_{RI}(s)$ be

$$
{}^{2}T_{RI}(s) = \begin{bmatrix} \dfrac{(s+1)^2\ (s+2)}{(3s+4)\ (s+1) + s(s+2)} \\[2ex] \dfrac{s(s+1)\ (s+2)}{(3s+4)\ (s+1) + s(s+2)} \end{bmatrix} \qquad \ldots(6.2.24)
$$

$$
\therefore\ U_2(s) = {}^{2}T_{RI}(s)\ y(s)
$$

$$
= \begin{bmatrix} \dfrac{(s+1)^2\ (s+2)}{(3s+4)\ (s+1) + s(s+2)} \\[2ex] \dfrac{s(s+1)\ (s+2)}{(3s+4)\ (s+1) + s(s+2)} \end{bmatrix} \begin{bmatrix} \dfrac{1}{(s+1)^2} + \dfrac{3s+4}{s(s+1)\ (s+2)} \end{bmatrix}
$$

$$
= \begin{bmatrix} \dfrac{(s+2) + \dfrac{s+1}{s}(3s+4)}{(3s+4)\ (s+1) + s(s+2)} \\[2ex] \dfrac{\dfrac{s(s+2)}{s+1} + 3s+4}{(3s+4)\ (s+1) + s(s+2)} \end{bmatrix}
$$

$$
= \begin{bmatrix} \dfrac{1}{s} \\[2ex] \dfrac{1}{s+1} \end{bmatrix} \qquad \ldots(6.2.25)
$$

$$
\therefore\quad u_2(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} \qquad \ldots(6.2.26)
$$

Thus $u_1(t)$ and $u_2(t)$ of 6.2.22 and 6.2.26 give the same output.

$$\therefore \quad U_1(s) = \begin{bmatrix} \dfrac{0.5(s+1)\ (s+2)}{3s+4} \\[2ex] 0.5(s+1) \end{bmatrix} \left[\frac{1}{s} \left\{ \frac{3s+4}{(s+1)(s+2)} \right\} + \frac{1}{(s+1)^2} \right]$$

$$= \begin{bmatrix} \dfrac{0.5}{5} + \dfrac{0.5(s+2)}{(1+s)\ (3s+4)} \\[2ex] \dfrac{0.5(3s+4)}{s(s+2)} + \dfrac{0.5}{1+s} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{0.5}{s} + \dfrac{0.5}{1+s} - \dfrac{1}{3s+4} \\[2ex] \dfrac{1}{s} + \dfrac{0.5}{s+2} + \dfrac{0.5}{1+s} \end{bmatrix} \qquad \ldots(6.2.2)$$

$$\begin{bmatrix} 0.5 + 0.5\ e^{-t} - \frac{1}{3}\ e^{-\frac{4}{3}t} \\[2ex] 1 + 0.5\ e^{-2t} + 0.5\ e^{-t} \end{bmatrix} \qquad \ldots(6.2.22)$$

It can be verified that

$y(s) = T(s)\ U_1(s)$

$$\begin{bmatrix} \dfrac{3s+4}{(s+1)\ (s+2)} & \dfrac{1}{s+1} \end{bmatrix} \begin{bmatrix} \dfrac{0.5}{s} + \dfrac{0.5}{1+s} - \dfrac{1}{3s+4} \\[2ex] \dfrac{1}{s} + \dfrac{0.5}{s+2} + \dfrac{0.5}{1+s} \end{bmatrix}$$

$$= \frac{1}{(s+1)^2} + \frac{3s+4}{s(s+1)(s+2)} \qquad \ldots(6.2.23)$$

Let another choice of $T_{RI}(s)$ be

$$ {}^{2}T_{RI}(s) = \begin{bmatrix} \dfrac{(s+1)^2\,(s+2)}{(3s+4)\,(s+1) + s(s+2)} \\[2ex] \dfrac{s(s+1)\,(s+2)}{(3s+4)\,(s+1) + s(s+2)} \end{bmatrix} \qquad \ldots(6.2.24) $$

$$ \therefore\ U_2(s) = {}^{2}T_{RI}(s)\, y(s) $$

$$ = \begin{bmatrix} \dfrac{(s+1)^2\,(s+2)}{(3s+4)\,(s+1) + s(s+2)} \\[2ex] \dfrac{s(s+1)\,(s+2)}{(3s+4)\,(s+1) + s(s+2)} \end{bmatrix} \begin{bmatrix} \dfrac{1}{(s+1)^2} + \dfrac{3s+4}{s(s+1)\,(s+2)} \end{bmatrix} $$

$$ = \begin{bmatrix} \dfrac{(s+2) + \dfrac{s+1}{s}(3s+4)}{(3s+4)\,(s+1) + s(s+2)} \\[2ex] \dfrac{\dfrac{s(s+2)}{s+1} + 3s+4}{(3s+4)\,(s+1) + s(s+2)} \end{bmatrix} $$

$$ = \begin{bmatrix} \dfrac{1}{s} \\[2ex] \dfrac{1}{s+1} \end{bmatrix} \qquad \ldots(6.2.25) $$

$$ \therefore\quad u_2(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} \qquad \ldots(6.2.26) $$

Thus $u_1(t)$ and $u_2(t)$ of 6.2.22 and 6.2.26 give the same output.

6.3 INPUT FUNCTION OBSERVABILITY

The input function observability is defined as the ability to decide uniquely the input to a given system during an interval when the corresponding output and initial conditions are given.

We have studied the case of the system when $p < m$. We have observed that the input corresponding to a given output (and initial conditions specified or assumed to be zero) is not unique. Thus the system cannot be input observable when $p < m$. Hence we can study the remaining alternatives $m \leq p$. Again as described previously taking into account the initial conditions properly we can write

$$y(s) = T(s)\, u(s) \qquad \ldots(6.3.1)$$

where $y(s)$ is known.

If we multiply 6.3.1 by a $T_{LI}(s)$ where

$$T_{LI}(s)\, T(s) = I_m \qquad \ldots(6.3.2)$$

We get

$$\begin{aligned} T_{LI}(s)\, y(s) &= T_{LI}(s)\, T(s)\, u(s) \\ &= u(s) \end{aligned} \qquad \ldots(6.3.3)$$

Let us suppose $u_1(s)$ causing $y(s)$ is known. So

$$y(s) = T(s)\, u_1(s) \qquad \ldots(6.3.4)$$

We want to reproduce this $y(s)$ by using an input say $u_2(s)$ such that

$$u_2(s) = {}^2T_{LI}(s)\ y(s) \qquad \ldots(6.3.5)$$

where ${}^2T_{LI}(s)$ is arbitrarily determined to satisfy 6.3.2.

Then multiplying both sides of 6.3.4 by ${}^2T_{LI}(s)$

$$\begin{aligned} {}^2T_{LI}(s)\ y(s) &= {}^2T_{LI}(s)\ T(s)\ u_1(s) \\ &= u_1(s) \end{aligned} \qquad \ldots(6.3.6)$$

So from 6.3.5 and 6.3.6,

$$u_1(s) = u_2(s) \qquad \ldots(6.3.7)$$

i.e. Independent of what $T_{LI}(s)$ we choose, the input obtained by 6.3.3 corresponding to given $y(s)$ is unique. It may be noted that the necessary and sufficient condition to satisfy 6.3.2 by some $T_{LI}(s)$ is to have $\rho\left[T(s)\right] = m$ ...(6.3.8)
It may be noted that for 6.3.8 to be true it is necessary that $m \leq p$. Thus the necessary and sufficient condition for a system to be input observable is that $\rho\left[T(s)\right] = m$ ...(6.3.9)

<u>Example 6.4</u> : A system is described by the state space triple,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \qquad \ldots(6.3.10)$$

i) Is it input observable?

ii) Find the input corresponding to a given output with

$$y(s) = \begin{bmatrix} \dfrac{2s+1}{s(s+1)^2} \\ \dfrac{2}{s(s+2)} \\ \dfrac{3s+1}{s(s+3)\ (s+1)} \end{bmatrix} \qquad \ldots(6.3.11)$$

iii) Verify that the input is unique independent of the choice of $T_{LI}(s)$

$p = 3$, $m = 2$ $\therefore$ $m < p$

$T(s) = C(sI-A)^{-1} B$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+2} & 0 \\ \frac{1}{s+3} & \frac{2}{s+3} \end{bmatrix} \qquad \ldots(6.3.12)$$

One $T_{LI}(s)$ can be defined as

$$^{1}T_{LI}(s) = \begin{bmatrix} 0 & 0.5(s+2) & 0 \\ 0 & -\frac{s+2}{4} & \frac{s+3}{2} \end{bmatrix} \qquad \ldots(6.3.13)$$

which satisfies $^{1}T_{LI}(s)\ T(s) = I_2$ ...(6.3.14)

Correspondingly for given $y(s)$

$$u_1(s) = T_{LI}(s)\ y(s) \qquad \ldots(6.3.15)$$

By substituting 6.3.13 and 6.3.11 in 6.3.15 we can get

$$u_1(s) = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+1} \end{bmatrix} \qquad \ldots(6.3.16)$$

$$\therefore \quad u_1(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} \qquad \ldots(6.3.17)$$

Making another choice of $T_{LI}(s)$ as

$$^{2}T_{LI}(s) = \begin{bmatrix} 2(s+1) & 0 & -(s+3) \\ (s+1) & -\dfrac{(s+2)}{2} & 0 \end{bmatrix} \qquad \ldots(6.3.18)$$

This gives us $^{2}T_{LI}(s).\ T(s) = I_m$

we have $u_2(s) = {}^{2}T_{LI}(s).y(s)$

$$\therefore u_2(s) = \begin{bmatrix} 2(s+1) & 0 & -(s+3) \\ (s+1) & -\dfrac{(s+2)}{2} & 0 \end{bmatrix} \begin{bmatrix} \dfrac{(2s+1)}{s(s+1)^2} \\ \dfrac{2}{s(s+2)} \\ \dfrac{3s+1}{s(s+3)(s+1)} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{2(s+1)}{s(s+1)} - \dfrac{3s+1}{s(s+1)} \\ \dfrac{2s+1}{s(s+1)} - \dfrac{1}{s} \end{bmatrix}$$

$$u_2(s) = \begin{bmatrix} \dfrac{1}{s} \\ \dfrac{1}{s+1} \end{bmatrix} \qquad \ldots(6.3.19)$$

and hence $u_2(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$ as expected $\qquad \ldots(6.3.20)$

Exercize 6 :

6.1 : Prove that whenever a system is output function controllable it is necessarily pointwise output controllable, i.e. whenever (6.2.4) is satisfied, (6.1.1) is necessarily satisfied.

6.2 : Determine the controllability, observability, output controllability (pointwise and functional) input observability and stability of the following systems,

i)
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

ii)
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

iii)
$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

iv)
$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

v)
$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

vi) Minimal realisation of

$$T(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} & \frac{1}{s^2+2s+1} \\ \frac{1}{s-1} & \frac{1}{s+3} & \frac{2}{s+1} \\ \frac{s}{s^2-1} & \frac{2s+7}{s^2+4s+3} & \frac{3}{s^2+2s+1} \end{bmatrix}$$

6.3 : Determine the input function required to reproduce the output

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1+e^{-t} - e^{-3t} \\ 1-e^{-2t} \end{bmatrix}.$$

in a system with transfer matrix

$$T(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ 1 & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix}$$

Is the input unique?

6.4 : Determine the input function required to get the output,

$$y(t) = 2 - 1.5e^{-t} - 0.5e^{-3t}$$

from a system with transfer matrix,

$$T(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+3} \end{bmatrix}$$

Is the input unique?

6.5 : Construct a system to be connected in cascade with the given system in order to solve the problem 6.3. Such a system is called a left inverse(w-3).

6.6 : Construct a system required to solve the problem 6.4. Such a system is called a right inverse(w-3).

CHAPTER - 7

TIME DOMAIN REALISATIONS

7.1 INTRODUCTION

There are three methods of representation as mentioned in the introduction to Chapter-2. Because each of the state space, differential operator and transfer matrix representations is specially suitable for particular techniques of design and analysis, it is many times necessary to transfer the system description from one form to another. This and the eighth chapter discuss the techniques of such transfers between different forms of representation. The transfer from S.S. and D.O. representations to T.M. and from T.M. to D.O. are straightforward and easy. The remaining three transfers require detailed discussion. Transfer from T.M. to either D.O. or S.S. is transferring from frequency domain to time domain. These transformations described in this chapter are called time domain realisations. As the time domain form corresponding to given T.M. is not unique, the problems of minimal realisation and choice of cononical form of S.S. representation arise and demand discussion.

The algorithms of realisation into multivariable controllable and observable companion forms are explained. Further, reduction of the same to minimal S.S. form is also derived. Controllable, observable and minimal realisations in the differential operator form are given at the end of this chapter.

7.2 Transfer Between Transfer Matrix and State Space Representations

A typical transfer matrix representation is described by the matrix equation.

$$y(s) = T(s)\ u(s) \qquad \ldots(7.2.1)$$

where T(s) is a p x m transfer matrix. For practical systems, T(s) is a proper transfer matrix i.e. every element of T(s) is a transfer function with degree of numerator less than or equal to that of the denominator. It can be observed that any such system represented by a proper transfer matrix T(s), can be represented by an equivalent state space quadruple (A, B, C, E). The laplace transformed output vector can be written in terms of these matrices as

$$y(s) = C(sI-A)^{-1}\ x(o) + [C(sI-A)^{-1}\ B + E]\ u(s) \qquad \ldots(7.2.2)$$

Initial conditions are assumed to be zero in the transfer matrix representation. Hence comparing 7.2.1 and 7.2.2 with x(o) = 0, we get the p x m matrix

$$T(s) = C(sI-A)^{-1}\ B + E \qquad \ldots(7.2.3)$$

This relation is the basic transfer relation to obtain the quadruple (A, B, C, E) equivalent to the given transfer matrix T(s). The first term of r.h.s. is a strictly proper matrix, i.e. the degree of each element of C $(sI-A)^{+}$ B is smaller than the degree of $|sI-A|$ which is the common denominator

of all elements. Thus if the given $T(s)$ is a strictly proper transfer matrix, E will be zero in the equivalent state space representation. Otherwise

$$E = \lim_{s \to \infty} T(s) \qquad \ldots(7.2.4)$$

i.e. the elements of E are non-zero only wherever the corresponding transfer function element of $T(s)$ has numerator and denominator of the same degree. In that case each element of E is equal to the ratio of the coefficients of the highest power terms of numerator and denominator, of the corresponding transfer function element of $T(s)$.

The relation 7.2.3 demands that the no. of rows of C must be equal to the no. of rows of $T(s)$ viz. equal to p The no. of columns of B must be equal to the no. of columns of $T(s)$ i.e. equal to m. The dimension of E must be the same as $T(s)$ i.e. $p \times m$. But there is no restriction on the dimension n of the square matrix A. Thus the equivalent system can have any dimension $n = n_1, n_2, n_3 \ldots$ etc. Obviously the state space representation corresponding to a given $T(s)$ is not expected to be unique with respect to its order. Moreover any state space representation can have an infinite no. of equivalent representations of same order satisfying the relation

$$\bar{x} = Qx \qquad \ldots(7.2.5)$$

where Q is an equivalent transformation matrix of dimension

n x n, as defined earlier in Chapter-2. It has been seen that for such equivalent representations (A, B, C, E) and $(\bar{A}, \bar{B}, \bar{C}, \bar{E})$ the transfer matrix is the same. So, for a given matrix $T(s)$ we can find

$$
\begin{aligned}
T(s) &= C(sI-A)^{-1} B + E \\
&= \bar{C}\,(sI-\bar{A})^{-1}\,\bar{B} + \bar{E}
\end{aligned}
\qquad \ldots(7.2.6)
$$

where A and $\bar{A}$ are related by

$$\bar{A} = Q\, A\, Q^{-1} \qquad \ldots(7.2.7)$$

Thus the state space representation corresponding to a given transfer matrix is not unique either with respect to its order or with respect to the set of state variables. This can be illustrated by the following example.

Example 7.1 :

Let the given transfer matrix $T(s)$ of dimension p x m be

$$T(s) = \frac{s}{s^2 + s + 1} \qquad \text{with } p = m = 1$$

The following two state space models realise this transfer matrix.

i)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -2 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

ii) $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

It can be easily verified that (i) and (ii) have the same transfer matrix equal to the one given. It may be noted that the two systems are of different orders. The sets of state variables are obviously different.

7.3 Minimal Realisation

Let us now study the transfer between T(s) and (A, B, C, E) from the point of view of controllability and/or observability of the system. Relation 7.2.6 can be written as

$$T(s) = \frac{R(s)}{p(s)} = \frac{C(sI-A)^{+} B + E|sI-A|}{|sI-A|} \qquad \ldots(7.3.1)$$

where R(s) is p x m polynomial matrix and p(s) is a monic polynomial which is the l.c.m. of the denominators of T(s). The degrees of p(s) and $|sI-A|$ are the same if there is no cancellation of any factor of $|sI-A|$ with each of the elements of $C(sI-A)^{+}B$. Obviously this case of no cancellation corresponds to minimum degree of $|sI-A|$. No cancellation also corresponds to system observability and controllability. Thus a minimal system is both observable and controllable.

However if there is any cancellation in the numerator and denominator of $C(sI-A)^{-1}B$, it need not always be due to either the unobservable or uncontrollable poles. There can be cancellation of some factors between $(sI-A)^{+}$ and $|sI-A|$. In that case A is called a noncyclic matrix. A matrix with

distinct eigenvalues is always cyclic. But the converse is not true (G.1). If an observable and controllable system has noncyclic A matrix then the deg. of $|sI-A|$ will be larger than deg. of $p(s)$. Thus the minimal system order will be larger than the degree of $p(s)$.

If over and above the cancellations in $(sI-A)^+$ and $|sI-A|$, there are any cancellations in $C(sI-A)^+$ and $|sI-A|$ or in $(sI-A)^+B$ and $|sI-A|$, the cancelling factors are called output decoupling zeros and input decoupling zeros respectively (R-1). These zeros are respectively equal to the unobservable poles and uncontrollable poles cancelling from $|sI-A|$. A system having such cancellations only is nonminimal.

Suppose (A, B, C, E) is any realisation of given T(s). Let it be uncontrollable. Then it can be converted, by equivalent transformation, to the uncontrollable companion form of 4.4.23 and 4.4.25, for which

$$T(s) = C(sI-A)^{-1} B + E$$

$$= C \begin{bmatrix} sI_{\bar{n}} - A_c & -A_n \\ 0 & sI_q - A_{uc} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + E \qquad \ldots(7.3.2)$$

where $q = n - \bar{n}$

$$\text{Then } T(s) = C \begin{bmatrix} [sI_{\bar{n}} - A_c) & B_c \\ 0 & \end{bmatrix} + E$$

$$= C_{\bar{n}} [sI_{\bar{n}} - A_c] B_c + E \qquad \ldots(7.3.3)$$

Thus (A_c, B_c, C_Π, E), corresponding to the controllable parts A_c and B_c of A and B respectively, is also a realisation of T(s) and is of smaller order than (A, B, C, E). The realisation (A_c, B_c, C_Π, E) can further be tested for observability. If it is unobservable, it can be converted to the unobservable companion form and the reduced order realisation corresponding to the observable part can be obtained. This realisation is minimal as it is both controllable and observable.

If we can realise the given T(s) by an observable or a controllable realisation (A, B, C, E) one step of the above mentioned order reduction procedure gets eliminated. A method of converting given T(s) into a controllable quadruple (A, B, C, E) is given in the next section.

Example 7.2 :

The pair (A, B) of the example 7.1 is in the uncontrollable companion form. Hence T(s) can be realised by the quadruple $[A_c, B_c, C_c, 0]$, where

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix} \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C_c = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

This is a controllable system. The realisation would be minimal if it is observable also. By applying the known test, it is found that $[A_c, B_c, C_c, 0]$ is not observable. It can be transformed into the equivalent unobservable companion form as

$$\bar{A} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & +1 & +1 \\ +1 & +1 & +1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and $$\bar{B} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$\bar{C} = C\,Q^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Thus $(\bar{A}, \bar{B}, \bar{C}, 0)$ are in the unobservable companion form. The observable parts of these give the realisation $(A_o, B_o, C_o,$ given by

$$A_o = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$B_o = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_o = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

This is the minimal realisation of $T(s)$.

7.4 Controllable Realisation Algorithm

As the transfer matrix representation retains only the observable and controllable modes, it is always possible to obtain the minimal realisation of any given transfer matrix T(s). But the minimal realisation cannot be obtained directly. Any other realisation obtained from T(s) can however be reduced to the minimal one by procedure described in the previous section. This section gives an algorithm for obtaining a controllable realisation of given T(s). This can further be reduced to minimal form. If an observable realisation is desired, the given T(s) can be transposed and the same algorithm for controllable realisation can be applied to $T^T(s)$. The resulting (A, B, C, E), when transposed, give the observable realisation (A^T, C^T, B^T, E), of T(s).

Thus once we have an algorithm for converting given T(s) into an equivalent controllable S.S. system (A, B, C, E), we are equipped to have either controllable, observable or minimal realisation of the given T(s). Following is a procedure to obtain (A, B, C, E) realisation, with A and B in the multivariable controllable companion form for the given T(s). Obviously the realisation would be controllable. Let T(s) be the given proper rational transfer matrix. Then the p x m matrix E can be obtained using relation 7.2.4. Then from 7.2.3 we have to determine (A, B, C) satisfying the equation

$$T(s) - E = C(sI-A)^{-1}B \qquad \ldots(7.4.1)$$

Let us assume that T(s) is expressed in the form $R(s)\, P^{-1}(s)$

with $R(s)$ and $P(s)$ as polynomial matrices. Then (A, B, C) must satisfy, from 7.4.1,

$$\left[R(s) - E\,P(s)\right] P^{-1}(s) = C\,(sI-A)^{-1}B \qquad \ldots(7.4.2)$$

The right hand sides of 7.4.1 and 7.4.2 represent a strictly proper matrix. That is, in each transfer function element of this matrix, the numerator has degree smaller than that of the denominator. Then referring to 2.7.4, we can write for the l.h.s. of 7.4.2,

$$d\left[P(s)\right]_{*j} > d\left[R(s) - E\,P(s)\right]_{*j} \qquad \ldots(7.4.3)$$

for $j = 1, 2, \ldots m$.

Let us use the notation

$$d\left[P(s)\right]_{*j} = d_j \text{ for } j = 1,2, \ldots m \text{ and}$$

$$\sum_{j=1}^{m} d_j = n \qquad \ldots(7.4.4)$$

Then

$$R(s) - E\,P(s) = M\,S(s) \qquad \ldots(7.4.5)$$

with a $p \times n$ constant matrix M and an $n \times m$ matrix

$$S(s) = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ s & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ s^{d_1-1} & 0 & \vdots & \vdots \\ 0 & 1 & \vdots & \vdots \\ \vdots & s & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & s^{d_2-1} & \vdots & \vdots \\ \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & s^{d_m-1} \end{bmatrix} \qquad \ldots(7.4.6)$$

with d_j of 7.4.4.

Thus from 7.4.2 and 7.4.5

$$C(sI_n-A)^{-1} B P(s) = M S(s) \qquad \ldots(7.4.7)$$

Let us choose the p x n matrix

$$C = M \qquad \ldots(7.4.8)$$

as C does not have any specific form in the controllable realisation (A, B, C, E). Now the matrices A and B which are assumed to be in the controllable companion form of 4.4.7 and 4.4.8 can be chosen to satisfy,

$$(sI_n-A)^{-1} B P(s) = S(s) \qquad \ldots(7.4.9)$$

i.e. $(sI_n-A) S(s) = B P(s)$ ...(7.4.10)

Let P(s) be written in the form

$$P(s)=\gamma_c D(s) - K S(s) \qquad \ldots(7.4.11)$$

with S(s) of 7.4.6 and with

$$D(s)=\begin{bmatrix} s^{d_1} & 0 & . & 0 \\ 0 & s^{d_2} & . & . \\ . & . & . & 0 \\ 0 & . & 0 & s^{d_m} \end{bmatrix} \qquad \ldots(7.4.12)$$

with d_j of 7.4.4. γ_c and K are constant matrices of appropriate dimensions. Substituting 7.4.11 in 7.4.10,

$$(sI_n-A) \; S(s) = B\gamma_c D(s) - BK S(s) \qquad \ldots(7.4.13)$$

The assumed structure of A has m diagonal blocks A_{ii} for

$i = 1$ to m. Let the dimension of A_{ii} be $d_i \times d_i$ with d_i equal to d_j of 7.4.4 for $i = 1$ to m. Taking into account the particular forms of A, B and S(s), it may be noted that all the rows of both sides of 7.4.13 except the m, σ_kth rows for $K = 1,2,\ldots m$, are identically equal to zero. Thus satisfying 7.4.13 amounts to choosing elements in these m significant rows of A and B such as to equate the corresponding rows on both sides of 7.4.13. The resulting m equations can be written as,

$$[sI_n - A]_m \, S(s) = B_m \gamma_c D(s) - B_m K \, S(s) \qquad \ldots(7.4.14)$$

where the suffix m denotes the concerned matrix formed by taking its m significant rows. Thus $(sI_n - A)_m$ is of dimension $m \times n$ and B_m is of dimension $m \times m$.

7.4.14 can be written as,

$$[sI_n]_m \, S(s) - [A]_m \, S(s) = B_m \gamma_c D(s) - B_m K \, S(s) \qquad \ldots(7.4.15)$$

But
$$[sI_n]_m \, S(s) = D(s) \qquad \ldots(7.4.16)$$

as can be verified directly.

So
$$[B_m \gamma_c - I_m] \, D(s) = [B_m K - A_m] \, S(s) \qquad \ldots(7.4.17)$$

The above equation can be satisfied by choosing

$$B_m \gamma_c = I_m$$

i.e.
$$B_m = \gamma_c^{-1} \qquad \ldots(7.4.18)$$

and $A_m = B_m K = \gamma_c^{-1} K$ ...(7.4.19)

Note, however, that the choice of B_m as in 7.4.18 requires that γ_c should be invertible and should be in the upper right triangular form as B of the assumed controllable companion form has corresponding B_m in the upper right triangular form. This requirement makes it necessary that P(s) should have highest degree terms of any column in the upper right triangle including the diagonal. Secondly P(s) should be column proper i.e. by definition, when expressed as in 7.4.11, its γ_c should be nonsingular. It is shown below that a proper tr. matrix T(s) can always be expressed as $R(s)\,P^{-1}(s)$ with P(s) column proper.

If we decompose the given T(s) into say $\bar{R}(s)\,\bar{P}^{-1}(s)$ where the $\bar{P}(s)$ is not column proper, it can be converted into a column proper matrix by simple column operations. That is we can write

$$\bar{P}(s)\, U_R(s) = P(s) \qquad \text{...(7.4.20)}$$

where P(s) is column proper.

$U_R(s)$ required to satisfy 7.4.20 can be obtained as described below.

If $d_1, d_2, \ldots d_m$ are the degrees of the columns of $\bar{P}(s)$, d_r is the highest of them and γ_c is singular, determine the constants $\alpha_1, \alpha_2 \ldots \alpha_m$ such that

$$\underset{m \times m}{[\gamma_c]}\ \underset{m \times 1}{[\alpha]} = [0] \qquad \text{...(7.4.21)}$$

Then $U_R(s)$ can be chosen to have its r^{th} column with the elements $\alpha_i\ s^{d_r - d_i}$ and remaining columns equal to the columns of unity matrix. To illustrate this method of converting given nonsingular polynomial matrix into column proper form, consider the following example.

Example 7.3 :

$$\begin{bmatrix} s^2-3 & s \\ 4 & 1 \end{bmatrix} \quad \text{with } |P(s)| = s^2-4s-3 \neq 0$$

$$\text{and } \gamma_c \text{ of } \overline{P}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{with } |\gamma_c| = 0$$

$d_1 = 2 \quad d_2 = 1$ hence $d_r = d_1 = 2$

$\alpha_1 = 1$ and $\alpha_2 = -1$

so that $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then $U_R(s) = \begin{bmatrix} \alpha_1 & s^{d_1-d_1} & 0 \\ \alpha_2 & s^{d_1-d_2} & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$$

Verify that

$$P(s) = \overline{P}(s)\ U_R(s)$$

$$= \begin{bmatrix} s^2-3 & s \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & s \\ 4-s & 1 \end{bmatrix}$$

is column proper. Further column operations can modify P(s) to give $\gamma_c(P(s))$ in the upper right triangular form. The decomposition $\bar{R}(s)\,\bar{P}^{-1}(s)$ of given T(s) with $\bar{P}(s)$ is non-column proper form can then be modified to

$$T(s) = R(s)\,P^{-1}(s) \qquad \ldots(7.4.22)$$

with $P(s) = \bar{P}(s)\,U_R(s)$, in column proper form

and $R(s) = \bar{R}(s)\,U_R(s)$ ...(7.4.23)

The steps of the algorithm for realising the given proper transfer matrix T(s) into a controllable S.S. form (A, B, C, E) with A and B in the controllable companion form can now be summarised as follows. Let this algorithm for frequency to time domain transfer into a controllable form be denoted by FT_c.

<u>Algorithm FT_c</u>

i) $\lim_{s \to \infty} T(s) = E$ ...(7.4.24)

ii) $T(s) = R(s)\,P^{-1}(s)$

with P(s) in column proper form and $\gamma_c[P(s)]$ in upper right triangular form.

iii) $P(s) = \gamma_c\,D(s) - K\,S(s)$

iv) $B_m = \gamma_c^{-1}$

v) $A_m = \gamma_c^{-1}\,K$

vi) $R(s) - E\,P(s) = M\,S(s)$

vii) $C = M$

Note that from (i) to (vii) we can write

viii) $P(s) = B_m^{-1}\,[D(s) - A_m\,S(s)]$ ...(7.4.25)

$R(s) = C\,S(s) + E\,P(s)$

These relations can be used as standard conversion formulae between frequency domain and time domain.

One way of writing any rational proper transfer matrix T(s) in the form R(s) $P^{-1}(s)$ with P(s) in the column proper form and $\gamma_c^{-1}[P(s)]$ in the form of B_m is as follows.
Let the i.jth element of T(s) be

$$T_{ij}(s) = \frac{r_{ij}(s)}{p_{ij}(s)} \qquad \ldots(7.4.26)$$

Let $p_j(s)$ = monic polynomial which is l.c.m. of all $p_{ij}(s)$.
Then 7.4.25 can be written as

$$T_{ij}(s) = \frac{R_{ij}(s)}{p_j(s)} \qquad \ldots(7.4.27)$$

Then T(s) can be written in the form R(s) $P^{-1}(s)$ where R(s) has elements $R_{ij}(s)$ and P(s) is m x m diagonal matrix with diagonal elements $p_j(s)$ for j = 1 ... m. Note that in this case γ_c is I_m. Hence $\gamma_c^{-1} = I_m = B_m$ is a particular case of general form of B_m. Choice of this particular form makes inversion of P(s) easy.

Example 7.4

Given

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Find the corresponding transfer matrix. The transfer matrix can be found by the relation,

$$T(s) = C(sI-A)^{-1}B + E$$

But this needs inversion of a 3 x 3 matrix. Easier method is to use relations viii of the algorithm FT_c, to give

$$P(s) = B_m^{-1}\left[D(s) - A_m S(s)\right]$$

$$B_m = I_2 = B_m^{-1} \quad d_1 = 2 \quad \text{and} \quad d_2 = 1$$

$$\therefore \quad D(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} \qquad S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad P(s) = \begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s^2-s-2 & 0 \\ 0 & s-2 \end{bmatrix}$$

$$R(s) = \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} P(s)$$

$$= \begin{bmatrix} s-2 & 2 \\ 2(s+1) & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ s^2-s-2 & s-2 \end{bmatrix} = \begin{bmatrix} s-2 & 2 \\ s^2+s & s-2 \end{bmatrix}$$

$$\therefore \quad T(s) = R(s)\, P^{-1}(s)$$

$$= \begin{bmatrix} \frac{s-2}{s^2-s-2} & \frac{2}{s-2} \\ \frac{s(s+1)}{(s-2)(s+1)} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s-2} \\ \frac{s}{s-2} & 1 \end{bmatrix}$$

Example 7.5

Obtain a state space controllable realisation of a system with transfer matrix,

$$T(s) = \begin{bmatrix} 2(s-1) & s+1 \\ 4 & -s \end{bmatrix} \begin{bmatrix} s+4 & 2(s+1) \\ 0 & s^2-s+4 \end{bmatrix}^{-1}$$

Let

$$R(s) = \begin{bmatrix} 2(s-1) & s+1 \\ 4 & -s \end{bmatrix}$$

and

$$P(s) = \begin{bmatrix} s+4 & 2(s+1) \\ 0 & s^2-s+4 \end{bmatrix}$$

$$d_1 = 1 \quad d_2 = 2 \quad \text{for } P(s)$$

$$\therefore \quad \gamma_c\,[P(s)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is nonsingular}$$

Thus P(s) is in the required form for applying FT_c. The realisation (A, B, C, E) can be obtained as follows :

i) $E = \lim_{s \to \infty} T(s) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

ii) $B_m = \gamma_c^{-1} = I_2$

$$\therefore \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

iii) $P(s) = \begin{bmatrix} s & 0 \\ 0 & s^2 \end{bmatrix} - K \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} s+4 & 2(s+1) \\ 0 & s^2-s+4 \end{bmatrix}$

$$\therefore \quad K = \begin{bmatrix} -4 & -2 & -2 \\ 0 & -4 & 1 \end{bmatrix} = A_m$$

$$\therefore A = \begin{bmatrix} -4 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \end{bmatrix}$$

iv) $R(s) - E\,P(s) = M\,S(s)$

$$\therefore \begin{bmatrix} 2(s-1) & s+1 \\ 4 & -s \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 2(s+1) \\ 0 & s^2-s+4 \end{bmatrix}$$

$$= M \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}$$

$$\therefore \quad M = \begin{bmatrix} -10 & -3 & -3 \\ 4 & 0 & -1 \end{bmatrix} = C$$

Thus quadruple (A, B, C, E) is determined.

7.5 Observable Realisation Algorithm

The easiest method to obtain an observable realisation of a given rational proper transfer matrix T(s) is to first get a controllable realisation of

$$(T(s) - E)^T = B^T (sI-A^T)^{-1} C^T , \qquad \ldots(7.5.1)$$

where (A^T, C^T) form a controllable pair. Then

$$T(s) - E = C(sI-A)^{-1} B \qquad \ldots(7.5.2)$$

and because $(A^T C^T)$ form a controllable pair, (A,C) form an observable pair. Thus (A, B, C, E) is an observable realisation of T(s).

Alternatively, an algorithm FT_o for transformation from frequency domain to observable system in time domain can be formed on lines similar to FT_c and given by,

Algorithm FT_o $\qquad \ldots(7.5.3)$

i) $\lim_{s \to \infty} T(s) = E$

ii) $T(s) = P^{-1}(s)\, Q(s)$

with P(s) row proper and with lower left triangular form of corresponding γ_r.

iii) $P(s) = D_o(s)\, \gamma_r - S_o(s)\, K$

where
$$D_o(s) = \begin{bmatrix} s^{d_1} & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & s^{dp} \end{bmatrix} \qquad \ldots(7.5.4)$$

is a p x p diagonal matrix

d_r ; $r = 1, 2 \ldots P$, is the degree of the r^{th} row of $P(s)$

and

$$S_o(s) = \begin{bmatrix} 1 & s & . & . & s^{d_1-1} & 0 & . & . & . & . & . & 0 \\ 0 & . & . & . & 0 & 1 & s^{d_2-1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 & 1 & . & s^{d_p-1} \end{bmatrix}$$

...(7.5.5)

is a p x n matrix.

iv) $C_p = \gamma_r^{-1}$ consists of the p significant columns of the C matrix in the observable companion form.

v) $A_p = K\,\gamma_r^{-1}$ consists of the p significant columns of the A matrix in the observable companion form.

vi) $Q(s) - P(s)\,E = S_o(s)M$

vii) $B = M$

(v), (vi), (vii) give

viii) $P(s) = \left[D_o(s) - S_o(s)\,A_p\right] C_p^{-1}$

and $Q(s) = S_o(s)B + P(s)\,E$ ...(7.5.6)

as the relations between the transfer matrix and observable s.s. description of a system.

If in the algorithms FT_c and FT_o, given $T(s)$ is not a proper transfer matrix it can be written as $T(s) = \overline{T}(s) + E(s)$, where $\overline{T}(s)$ is strictly proper and the algorithm can be applied to realise $\overline{T}(s)$ as $C\,(sI-A)^{-1}\,B$.

7.6 Minimal D.O. Realisation

Let the transfer matrix be

$$T(s) = R(s)\ P^{-1}(s) \qquad \ldots(7.6.1)$$

We can find out the g.c.r.d. of R(s) and P(s) as explained in section 5.4. Let it be $G_R(s)$.

Then $$T(s) = \overline{R}(s)\ \overline{P}^{-1}(s) \qquad \ldots(7.6.2)$$

where $$P(s) = \overline{P}(s)\ G_R(s)$$

and $$R(s) = \overline{R}(s)\ G_R(s) \qquad \ldots(7.6.3)$$

as $G_R(s)$ is the greatest common right divisor. $|\overline{P}(s)|$ is of the smallest possible degree to express given T(s) in the form 7.6.1. The system represented by the D.O. form,

$$\overline{P}(D)\ z(t) = U(t)$$
$$y(t) = \overline{R}(D) z(t) \qquad \ldots(7.6.4)$$

with the D.O. quadruple $(\overline{P}(D),\ I,\ \overline{R}(D),\ 0)$ is observable as any g.c.r.d. of $\overline{P}(D)$ and $\overline{R}(D)$ is unimodular. It is controllable as any g.c.l.d. of $\overline{P}(D)$ and Q(D) (=I) is obviously unimodular. Thus the D.O. realisation 7.6.4 is minimal.

For T(s) expressed as

$$T(s) = \overline{P}^{-1}(s)\ Q(s) \qquad \ldots(7.6.5)$$

We can write T(s) as

$$\underset{p \times m}{T(s)} = \underset{p \times p}{\overline{P}^{-1}(s)}\ \underset{p \times m}{\overline{Q}(s)} \qquad \ldots(7.6.7)$$

where

$$P(s) = G_L(s)\ \bar{P}(s)$$

and $$Q(s) = G_L(s)\ \bar{Q}(s) \qquad \ldots(7.6.8)$$

$G_L(s)$ being the g.c.l.d. of P(s) and Q(s).

Obviously deg. $|\bar{P}(s)|$ is min.

The corresponding system representation in D.O. form would be

$$\bar{P}(D)\ x(t) = \bar{Q}(D)\ u(t)$$

and $$z(t) = x(t) \qquad \ldots\ (7.6.9)$$

i.e. the D.O. quadruple is (P, Q, I, 0). The system is controllable and $\bar{P}(s)$ and $\bar{Q}(s)$ have unimodular g.c.l.d. and is observable also as $\bar{P}(D)$ and I also have unimodular g.c.r.d.

Example 7.6 :

Let

$$T(s) = \begin{bmatrix} s^2 & -1 \\ 0 & s^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s \\ s & 1 \end{bmatrix}$$

$$= P^{-1}(s)\ Q(s) \quad \text{say.}$$

Determine the corresponding minimal D.O. representation.

Given P(s) and Q(s) have their g.c.l.d. unimodular as shown in Example 5.1. Hence the minimal realisation is

$$P(D)\ x(t) = Q(D)\ u(t)$$

$$z(t) = x(t)$$

<u>Exercize 7</u> :

7.1 : The transfer matrix of a system is

$$T(s) = R(s)\, P^{-1}(s)$$

$$\begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} s^2 & 0 \\ -1 & s-1 \end{bmatrix}^{-1}$$

Obtain controllable realization in the state space form. Is the realization expected to be minimal? Verify in the time domain.

7.2 : Decompose the following transfer matrices into $R(s)$ and $P(s)$ such that :

i) $T(s) = R(s)\, P^{-1}(s)$

ii) $R(s)$ and $P(s)$ are relatively right prime

iii) $P(s)$ is column proper

a) $$T(s) = \begin{bmatrix} \dfrac{s+1}{s^2} & \dfrac{s+2}{s^2+1} \\[2ex] \dfrac{2}{s} & \dfrac{2s+3}{s^2+1} \end{bmatrix}$$

b) $$T(s) = \begin{bmatrix} \dfrac{(s-2)(s+1)}{s(s-1)^2} & \dfrac{1}{(s-1)^2} \\[2ex] -\dfrac{1}{s} & 0 \\[2ex] \dfrac{2}{s(s-1)} & \dfrac{1}{s-1} \end{bmatrix}$$

7.3 : How will you modify the decomposition of $T(s)$ of 7.1 in order to get the controllable realisations whose

subsystems are each controllable from a single input?

7.4 : The state space triple for a mv control system is given by :

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & -2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find the corresponding transfer matrix without calculating any matrix inverse.

7.5 : Obtain observable state space realisation for the T(s) of 7.2 (b). Is it minimal?

7.6 : Obtain minimal differential operator realisation of T(s) of problem 7.2 (a). Further convert the D.O. form to S.S. form.

7.7 : Decompose the T(s) of 7.1 into $P_o^{-1}(s)\, Q_o(s)$ with $P_o(s)$ and $Q_o(s)$ relatively left prime and $P_o(s)$ row proper. Find the minimal realisation. Determine the values of controllability index and observability index for the minimal realization.

7.8 : The state space triple (A, B, C) of a system is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculate the input, output decoupling zeros if any. Is the matrix A cyclic? Find out the transfer matrix T(s) and convert it into the Smith McMillan form.

CHAPTER - 8

TRANSFER FROM D.O. TO S.S. REPRESENTATION

8.1 Introduction

Many times the mathematical equations written to describe the system dynamics lead directly to the D.O. form of system representation given by 2.2.1 and repeated below :

a) $P(D)\ z(t) = Q(D)\ u(t)$

and b) $y(t) = R(D)\ z(t) + W(D)\ u(t)$ ...(8.1.1)

To convert this to the s.s. form of representation, following points of equivalence between the two forms of representation must be observed.

1) The number of initial conditions required in a st. sp. representation is equal to n, the no. of state variables. So the number of state variables x(t) satisfying the equation

$$\dot{x} = Ax + Bu \quad \text{...(8.1.2)}$$

should be equal to the number of initial conditions in 8.1.1. The initial conditions defined in 8.1.1 would be the initial values of the variables z(t) and their derivatives corresponding to given P(D). The no. of independent initial conditions for 8.1.1 is equal to the degree of the determinant of P(D), as shown below. Let the given P(D) be transformed into an upper right triangular form $\bar{P}(D)$ by simple row operations, such that

$$\bar{P}(D) = U_L\ (D)\ P(D) \quad \text{...(8.1.3)}$$

and the highest degree element of j^{th} column of $\overline{P}(D)$ with degree d_j occurs on the major diagonal for all j = 1 to q.

Hence $\deg |\overline{P}(D)| = \sum_{j=1}^{q} d_j = n$ ...(8.1.4)

From 8.1.3

$$|\overline{P}(D)| = |U_L(D)| \cdot |P(D)| \quad \text{...(8.1.5)}$$

$$\therefore \quad \deg |\overline{P}(D)| = \deg |P(D)| = n$$

as $\deg |U_L(D)| = 0$, $U_L(D)$ being a unimodular matrix.

The given system 8.1.1 can be represented as

$$U_L(D)\ P(D)\ z(t) = U_L(D)\ Q(D)\ u(t)$$

i.e. $\overline{P}(D)\ z(t) = \overline{Q}(D)\ u(t)$ ...(8.1.6)

The initial conditions to be defined for 8.1.6, being equal to the independent initial conditions in 8.1.1 are

$$\sum_{j=1}^{m} d_j = n = \deg |\overline{P}(D)| = \deg |P(D)| \quad \text{...(8.1.7)}$$

Thus the no. of state variables in the equivalent s.s. representation equal to n of 8.1.7. is the system order

2) The relation defining the equivalence between z(t) and x(t) should be such that it should be possible to determine x(t) uniquely corresponding to given z(t) and u(t). The relation of equivalence between x(t) and z(t) can be written in either of the following two forms :

a) $x(t) = L(D)\ z(t) + M(D)\ u(t)$

or b) $z(t) = C_0\ x(t) + H(D)\ u(t)$ ...(8.1.8)

It is obvious that (a) gives $x(t)$ uniquely for given $z(t)$ and $u(t)$. 8.1.8.b also defines $x(t)$ uniquely if for the corresponding s.s. pair (A,B), A and C_0 form an observable pair. In either case the dimension of the state vector is equal to the degree of $|P(D)|$.

Sections 8.2 and 8.3 describe two methods for performing the D.O. representation of 8.1.1 to s.s. representation with equivalence relation 8.1.8(a). Section 8.4 describes similar transformation with equivalence relation 8.1.8b.

8.2 Triangularisation Method :

Let $\overline{P}(D)$ be obtained as in 8.1.3. Correspongly let

$$\overline{Q}(D) = U_L(D)\, Q(D) \qquad \ldots(8.2.1)$$

so that 8.1.1 can be written as

$$\overline{P}(D)\, z(t) = \overline{Q}(D)\, u(t) \qquad \ldots(8.2.2)$$

with

$$\overline{P}(D) = \begin{bmatrix} P_{11}(D) & \cdot & \cdot & \cdot & P_{1q}(D) \\ 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & P_{qq}(D) \end{bmatrix} \qquad \ldots(8.2.3)$$

$P_{ii}(D)$ is the highest degree polynomial of i^{th} column with degree d_i for $i = 1$ to q. Hence 8.2.2 can be written as

$$\begin{bmatrix} P_{11}(D) & . & . & . & P_{1q}(D) \\ 0 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & 0 & P_{qq}(D) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ . \\ . \\ z_q \end{bmatrix} = \begin{bmatrix} Q_{1*}(D) \\ Q_{2*}(D) \\ . \\ . \\ Q_{q*}(D) \end{bmatrix} u \qquad \ldots(8.2.4)$$

The last one of the q equations of 8.2.4 can be written as

$$P_{qq}(D)z_q = Q_{q*}(D)u \qquad \ldots(8.2.5)$$

Let the highest degree element of $P_{qq}(D)$ be of degree $d_q = h$ and let the highest degree element of $Q_{q*}(D)$ be of degree L, less than h. Then 8.2.5 can be written as

$$p_h D^h(z_q) + p_{h-1}D^{h-1}(z_q) + \ldots + p_o\, z_q$$
$$= q_{L*}D^L(u) + q_{(L-1)*}D^{L-1}(u) + \ldots + q_o u \qquad \ldots(8.2.6)$$

where p_h is made equal to one and D^r denotes r^{th} derivative of the variable in the brackets. The $d_q = h$ state variables for 8.2.6 can be chosen as

$$x_1 = z_q$$
$$x_2 = p_{h-1}z_q + D(z_q)$$
$$x_3 = p_{h-2}z_q + p_{h-1}D(z_q) + D^2(z_q)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad .$$
$$x_{h-L+1} = p_L z_q + p_{l+1}D(z_q) + \ldots + D^{h-1}(z_q) - q_{1*}u$$
$$x_{h-L+2} = p_{L-1}z_q + p_1 D(z_q) + \ldots + D^{h-1+1}(z_q)$$
$$- q_{1*}D(u) - q_{(L-1)*}u$$

. . . . . .

$$x_h = p_1 z_q + p_2 D(z_q) + \dots + D^{h-1}(z_q)$$
$$- q_{1*}u - q_{2*}D(u) - \dots\dots - q_{h*}D^{h-1}(u) \qquad \dots(8.2.7)$$

Note that 8.2.7 is similar in form to 8.1.8a.

The state equations corresponding to 8.2.7 can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_h \end{bmatrix} = \begin{bmatrix} -p_{h-1} & 1 & 0 & \cdot & \cdot & 0 \\ -p_{h-2} & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ -p_o & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_h \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ q_{1*} \\ q_{o*} \end{bmatrix} u \qquad \dots(8.2.8)$$

Thus $h = d_q$ state equations can be written corresponding to q^{th} equation of 8.2.4. Similarly for any r^{th} equation of 8.2.4, $h = d_r$ state equations can be written for $r = q, q-1 \dots 1$, with d_r equal to the degree of the highest degree element of $P_{rr}(D)$. Thus the total number of state equations $= \sum_{r=1}^{r} d_r = n =$ the total number of initial conditions to be specified for z and its derivatives. Thus the order of the equivalent state space representation is equal to the degree of determinant of $\dot{P}(D)$.

Note that the first equation of each of the q sets similar to 8.2.7 defines $\overset{+}{z}$ explicitly as functions of x. Hence

the output equation in s.s. representation can be obtained by substituting for z in 8.1.1b.

<u>Example 8.1</u>

Consider the system

$$\begin{bmatrix} D(D+1) & D+2 & D+1 \\ 0 & (D+1)(D+2) & D \\ 0 & 0 & (D+2)(D+1)^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ D & 0 & 1 \\ D+3 & 1 & D \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

with P(D) in the required triangular form. The order of equivalent S.S. representation $= \sum_{r=1} d_r = 2 + 2 + 3 = 7$

The last equation is

$$\dddot{z}_3 + 4\,\ddot{z}_3 + 5\,\dot{z}_3 + 2\,z_3 = \dot{u}_1 + \dot{u}_3 + 3\,u_1 + u_2$$

Then the state equations, according to 8.2.8 corresponding to z_3 are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 1 \\ -5 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

The second equation in z is

$$(D^2 + 3D + 2)z_2 + Dz_3 = Du_1 + u_3$$

Substituting $z_3 = x_1$ from 8.15

$$(D^2 + 3D + 2)z_2 = -Du_1 + u_3 - Dx_1$$

$$= Du_1 + u_3 + 4x_1 - x_2$$

Substituting from the above state equations.

Treating x_1, x_2 as inputs in these equations, the state equations can be written using,

$$\begin{bmatrix} \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} u$$

Similarly from equation in z_1 corresponding to row 1, the remaining two state equations can be written as

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_6 \\ x_7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The complete set of 7 state equations can then be written as

$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 0 \\ 4 & -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 3 & -1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u$$

and

$$z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x$$

Using equations 8.2.7 the x can be expressed in the form of function of z and u as in 8.1.8a

8.3 Method for Proper Polynomial Matrix P(D)

The state equations corresponding to 8.1.1(a) can be written more directly if P(D) can be written as

$$P(D) = P_h D^h + P_{h-1} D^{h-1} + \dots P_1 D + P_o \qquad \dots(8.3.1)$$

with $P_h = I_q$ where h is the degree of the highest degree element of P(D). Such a P(D) is called a proper matrix. In this case the degree of the determinant of P(D) is qh. If P_h is not unit matrix, but is any nonsingular matrix then it can be transformed into a unit matrix by premultiplication of P(D) by an appropriate unimodular matrix, hence without introducing any change in the degree of its determinant. Then the n = qh state variables can be defined as

$$x_{1q} = z$$
$$x_{2q} = P_{h-1}\, z + D(z)$$
$$x_{3q} = P_{h-2}\, z + P_{h-1}\, D(z) + D^2(z)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_{(h-L+1)_q} = P_1\, z + P_{L+1}\, D(z) + \dots D^{h-1}(z) - Q_1\, u$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_{hq} = P_1\, z + P_2\, D(z) + \dots D^{h-1}(z)$$
$$- Q_1 u - Q_2 D(u) \dots - Q_L\, D^{L-1}(u) \qquad \dots(8.3.2)$$

with

$$\text{i)}\; Q(D) = Q_L\, D^L + Q_{L-1}\, D^{L-1} + \dots Q_0 \qquad \dots(8.3.3)$$

i.e. with L = the degree of the highest degree element of Q(D) which is smaller than h.

and ii) with x_{rq}, a q^{th} order state vector for r = 1 to h.

Similarity in equations 8.2.7 and 8.3.3 may be noted. The state equations corresponding to 8.3.3 can be written in the matrix form as,

$$\begin{bmatrix} \dot{x}_{1q} \\ \dot{x}_{2q} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{hq} \end{bmatrix} = \begin{bmatrix} -P_{h-1} & I & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & I \\ -P_0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} x_{1q} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{hq} \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ Q_1 \\ \cdot \\ Q_0 \end{bmatrix} u \qquad \dots(8.3.4)$$

Equations 8.2.7 and 8.3.2 can be written in the form of

equation 8.1.8(a), which gives unique $x(t)$ for given $z(t)$ i.e. also unique $x(o)$ for given initial conditions on $z(t)$ and its derivatives.

Example 8.2

Let the given D.O. form of system have

$$P(D) = \begin{bmatrix} D^2+2D+3 & D+2 \\ D+1 & D^2+1 \end{bmatrix}$$

$$Q(D) = \begin{bmatrix} D+1 & 2 \\ 1 & 2D+1 \end{bmatrix}$$

$$P_h = P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad P_o = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$Q_1 = Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad Q_o = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \dot{x}_{12} \\ \dot{x}_{22} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -3 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

8.4 Method for Row Proper Matrix P(D)

Let P(D) be row proper. If not, assume that it is transformed by appropriate $U_L(D)$ such that it is row proper and its γ_r is in the lower left triangular form. Then

$$P(D)\ z(t) = Q(D)\ u(t) \qquad \ldots(8.4.1)$$

can be transformed by the observable algorithm FT_Q to a form say,

$$\dot{x} = Ax + Bu \qquad \ldots(8.4.2)$$

and $$z = C_o\, x \qquad \ldots(8.4.3)$$

So that

$$C_o(sI-A)^{-1}B = P^{-1}(s)\, Q(s) \qquad \ldots(8.4.4)$$

This is possible obviously if $P^{-1}(s)\, Q(s)$ is strictly proper. Hence we make the assumption that

$$d_r\,[P(s)] > d_r\,[Q(s)] \qquad \ldots(8.4.5)$$

i.e. degree of any row of $P(s)$ is larger than the degree of the corresponding row of $Q(s)$, the degree of a row being defined as the degree of the highest degree element of that row. We have made the same assumption in the previous sections 8.2 and 8.3.

If the given D.O. representation of system is such that 8.4.5 is not satisfied, following modification can be used before using any of the three methods described above. Let the given representation have

$$\overline{P}(D)\; z = \overline{Q}(D)u \qquad \ldots(8.4.6)$$

with $\overline{P}^{-1}(D)\,\overline{Q}(D)$, not a strictly proper matrix. Write the transfer matrix, $\overline{P}^{-1}(D)\,\overline{Q}(D)$ in the form,

$$\overline{P}^{-1}(D)\,\overline{Q}(D) = M(D) + H(D) \qquad \ldots(8.4.7)$$

where $M(D)$ is a strictly proper transfer matrix and $H(D)$ is a polynomial matrix.

Let $$M(D) = \overline{P}^{-1}(D)\, Q(D) \qquad \ldots(8.4.8)$$

Then

$$z(t) = \overline{P}^{-1}(D)\ \overline{Q}(D)\ u(t)$$
$$= M(D)\ u(t) + H(D)\ u(t)$$
$$= \overline{P}^{-1}(D)\ Q(D)\ u(t) + H(D)\ u(t) \qquad \ldots(8.4.9)$$

If we realise $\overline{P}^{-1}(D)\ Q(D)$ by $(A,\ B,\ C_o)$ using any of the above methods.

$$z(t) = C_o\ x(t) + H(D)\ u(t) \qquad \ldots(8.4.10)$$

If given $\overline{P}^{-1}(D)\ \overline{Q}(D)$ is a proper transfer matrix, $H(D) = E$, a constant matrix.

The equation 8.1.1(b), namely,

$$y(t) = R(D)\ z(t) + W(D)\ u(t) \qquad \ldots(8.4.11)$$

can be transformed to the form

$$y(t) = C\ x(t) + E(D)\ u(t) \qquad \ldots(8.4.12)$$

by substituting from 8.4.10 for $z(t)$ and substituting in the resulting equation from

$$\dot{x} = Ax + Bu \qquad \ldots(8.4.13)$$

for the derivatives of x.

Exercise 8 :

(8.1) The D.O. form description of a system is :

i) $\frac{d^2 z_1}{dt^2} + \frac{d^2 z_2}{dt^2} = -z_1 + 2u$

ii) $\frac{dz_1}{dt} + \frac{dz_2}{dt} = -Z_2 + u$

iii) $y_1(t) = z_1(t)$

iv) $y_2(t) = z_1 + 2z_2$

The initial conditions are :

$z_1(0) = (0) \qquad z_2(0) = 5$

$\dot{z}_1(0) = -5 \qquad \dot{z}_2(0) = 0$

a) What is the order of an equivalent S.S. representation?

b) Convert the given system description into an equivalent S.S. description.

(8.2) Prove that any polynomial matrix P(D) can be transformed to the form of $\overline{P}(D)$ of equation (8.1.5).

(8.3) The system description in the D.O. form is as in the illustrative example 8.2. Obtain the equivalent state space representation by the method of example 8.1.

(8.4) For P(D) of example 8.1 write :

i) the column degrees

ii) the row degrees

iii) the degree of P(D)

iv) $\gamma_c\,[P(D)]$

v) $\gamma_r\,[P(D)]$

Is the matrix P(D),

a) Column proper,

b) Row proper,

c) Proper?

(8.5) The system description in the D.O. form is as in example 8.2. Obtain the equivalent S.S. representation by the method of section 8.4. Are the S.S. systems obtained in the illustrative example 8.2 and in exercise problems 8.3 and 8.5 equivalent?

(8.6) Given

$$T(s) = \begin{bmatrix} s-2 & 2 \\ s^2+s & s-2 \end{bmatrix} \begin{bmatrix} s^2-s-2 & 0 \\ 0 & s-2 \end{bmatrix}^{-1}$$

Convert the system description to D.O. form first and then to S.S. form.

+ + +

CHAPTER - 9

ARBITRARY POLE ALLOCATION IN TIME DOMAIN

9.1 Introduction :

Previous chapters were devoted to analysis of multi-variable control systems. This and the following chapters in this last part of the book discuss the problems about design and methods of solving them.

The aim of the design is to modify the given system, inorder to give to it certain desired characteristics. The most important characteristic of desired performance is stability. As is well known this can be achieved in the linear systems by locating the system poles in l.h.s. of s-plane. The particular locations of the poles in l.h.s. can be defined by considering the desired modes of system response.

The change in the system pole locations, i.e. the allocation of system poles in desired positions, effectively requires an appropriate modification in the system matrix A in the state space representation or in the matrix P(s) in the frequency domain representation.

It may be noted that the desired system response in terms of the magnitudes and exact variation with respect to time of any output element cannot be translated directly and exactly into the desired pole locations. The given pole

locations only assure existence of the corresponding modes of response. The combination of different modes and the relative weightage of different factors is function of the numerator dynamics and innumerable possible combinations of initial conditions and inputs. Optimization of certain performance index as a design criterion leads to specifying the input as function of time for assumed terminal conditions or specifying input as function of state in the optimal feedback control systems. Optimization is not included in the scope of this book. Another desirable specification of the multivariable control system design is noninteractive design. Different degrees of decoupling between different sub-systems can be specified. Disturbance decoupling can be a specific requirement. Different analytical and graphical techniques in time and frequency domain are available to tackle the problem of decoupling.

A more stringent design specification would be to achieve a given desired system transfer matrix by compensation i.e. to match the system transfer matrix with a model transfer matrix. The problem of design for all such specifications and its solution using different design techniques are discussed in this part on design.

The problem of specified arbitrary pole location is discussed first. Depending on the choice of the form of representation of the system, different compensation techniques in time domain or frequency domain can be chosen to attain

the desired characteristic polynomial of the compensated system. Time domain techniques using S.S. representation will be discussed first and frequency domain techniques using T.M. representation will be discussed later.

While attaining the desired pole location it is necessary to see its effect on the controllability and observability which are desirable properties. Hence the canonical forms revealing these properties are convenient to start the design with and obviously the design effort can be directed towards keeping these forms unchanged by compensation.

The time domain design techniques will mainly consist of linear state variable feedback (l.s.v.f.) and linear output feedback (l.o.f.) and allied topics like state estimation and design and realization of observers. The frequency domain techniques are based on modifying the system transfer matrix by feedback and/or feed forward compensation.

In this chapter, we confine to the linear state variable feedback (l.s.v.f.). The importance of l.s.v.f. is that it is proved that the solution of a regulator type optimal problem is a l.s.v.f. The simplicity is in the fact that the feedback matrix is a constant matrix. Hence this is easy to realise provided full state is measurable. Its quality is in that it can achieve any arbitrary pole allocation for a controllable system as proved in what follows.

9.2 Linear State Variable Feedback in Controllable Systems

The block diagram of the system compensated by l.s.v.f. is drawn below :

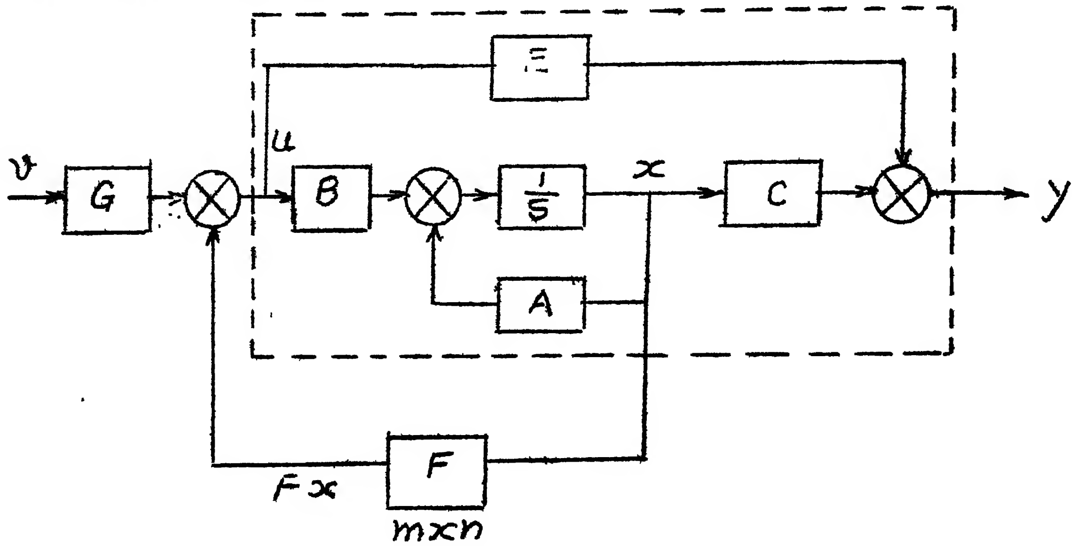

Fig. 9.2.1

The original uncompensated system is enclosed by dotted line. Its equations are

$$\dot{x} = Ax + Bu$$

$$y = Cx + Eu \qquad \ldots(9.2.1)$$

u = ' when there is no feedback. In many practical systems the linear description of system dynamics of the form 9.2.1 can be obtained only by purturbation technique from the original nonlinear dynamics. In that case x represents the deviation from the desired operating values of corresponding system variables. Naturally in the steady state desired value of x is zero and hence in such case the reference input v is zero. The system input vector u with l.s.v.f. of Fig. 9.2.1 is

$$u = Gv + Fx \qquad \ldots(9.2.2)$$

and system equation 9.2.1 becomes

$$\dot{x} = Ax + BFx + BGv$$
$$y = Cx + EFx + EGv \qquad \ldots(9.2.3)$$

A is n x n and each of its elements is modified by BF. Thus the design problem is to modify n^2 elements of A by choice of mn elements of F in such a way as to get desired values of n coefficients of the characteristic polynomial

$$\Delta_d(s) = |sI-(A+BF)| \qquad \ldots(9.2.4)$$

corresponding to the desired closed loop pole locations. Let us assume the given (A,B) to be controllable and in the controllable companion form of 4.4.7 and 4.4.8. Then the only elements of A to be adjusted by the feedback are the mn significant elements of the m significant rows of A. As F has mn elements the decision about choice of these is unique corresponding to a particular choice of (A + BF) which can give desired $\Delta_d(s)$. The most convenient manner of determining desired (A + BF) corresponding to desired $\Delta_d(s)$ is to assume a scalar companion from 2.6.4 for A + BF, which has only n significant elements; equal to the coefficients of desired $\Delta_d(s)$; in the last row. It may be noted that because of the particular form of B, having all nonsignificant rows zero, the nonsignificant rows of A are not affected by adding BF to it. Thus the form of A + BF is also the mv controllable companion form. Hence the matrix A can be changed to scalar

controllable form by proper choice of elements of F, as scalar controllable form is a particular case of mv controllable form. The choice of mn elements of F is unique in this case also, as the m significant rows of the matrix A except the n th row are to be altered to the nonsignificant rows of the scalar controllable companion form and the n th row of A is to be altered to have coefficients of the desired characteristic polynomial.

The controllability of the compensated system depends on whether or not the pair (A + BF, BG) is controllable. As A + BF is in the scalar controllable form and $m > 1$, BG is required to have one of its columns in the scalar controllable companion form of 2.6.5 to make the compensated system controllable. B can be modified by column operation i.e. by post-multiplication by some elementary matrix G so that BG is having one of its columns equal to $\left[0 \;.\;.\;.\;.\; 0\;1\right]^T$.

From 9.2.1 the transfer matrix of the uncompensated system is

$$\begin{aligned} T(s) &= C(sI-A)^{-1}B \\ &= R(s)\,P^{-1}(s) \end{aligned} \qquad \ldots(9.2.5)$$

say, with R(s) and P(s) given by 7.4.25. From 9.2.3 the closed loop transfer matrix,

$$\begin{aligned} T_{cl}(s) &= (C + EF)\left[sI-(A + BF)\right]^{-1}BG \\ &= R_F(s)\,P_F^{-1}(s)G \end{aligned} \qquad \ldots(9.2.6)$$

$R_F(s)$ and $P_F(s)$ can be written, using 7.4.25 as

$$P_F(s) = [B_m]^{-1} [D(s) - (A + BF)_m S(s)]$$

$$= B_m^{-1} [D(s) - A_m S(s)]$$

$$- B_m^{-1} B_m F S(s)$$

$$= P(s) - F S(s) \qquad \ldots(9.2.7)$$

$$R_F(s) = (C + EF) S(s) + E P_F(s)$$

$$= C S(s) + E [P_F(s) + F S(s)] \qquad \ldots(9.2.8)$$

Therefore from 9.2.7

$$R_F(s) = C S(s) + E P(s)$$

$$= R(s) \qquad \ldots(9.2.9)$$

Thus the l.s.v.f. does not change the R(s) matrix, but changes P(s) matrix only. Note that

$$F(s) = P_F(s) - P(s)$$

$$= F S(s) \qquad \ldots(9.2.10)$$

9.3 L.S.V.F. in Uncontrollable Systems

From the previous section it is obvious that the poles of the controllable system can be located at arbitrary, desired, specified locations by proper choice of feedback matrix F.

If similar procedure is followed for uncontrollable system, i.e. the system description is converted to uncontrollable companion form, it can be easily noticed that the uncontrollable part of A cannot be affected by the elements of BF. This is because the last $n-\bar{n}$ rows of BF are zero and hence cannot affect the last $n-\bar{n}$ rows consisting of the

uncontrollable part of A. As stated already the characteristic polynomial of A is given by (4.4.26) i.e.

$$|sI_n - A| = |sI_{\bar{n}} - A_c| \cdot |sI_{n-\bar{n}} - A_{uc}| \qquad \ldots(9.3.1)$$

The only factor $|sI_{\bar{n}} - A_c|$ can be altered as specified. But the uncontrollable poles cannot be affected by l.s.v.f.

9.4 <u>Factors of Freedom in the Design Procedure</u>

a) We have, as the first step, converted the A and B matrices of given controllable system in multivariable controllable companion form. It is possible to write the companion form with different combinations of d_i such that $\sum_{i=1}^{r} d_i = n$ for $r \leq m$ (refer 4.4.20). This will lead to number of possible choices, the design procedure remaining the same otherwise. If the form of A chosen is such that $\sum_{i=1}^{r} d_i = n$ with $r < m$, the corresponding form of B is having r columns in the canonical form and remaining m-r columns having no specific form, as given by 4.4.21. If, in such case BF is not to change the form of A, then it is necessary to have first r rows of F only nonzero. That is the rank of F will then be $r < m$ and only rn elements of F are to be determined instead of mn elements. The extreme case is $r = 1$ when we can write the given matrix A into an equivalent scalar companion form. In that case F will be of rank one only.

Another major choice is about the form of (A + BF), which we have chosen as the scalar controllable form. This can alternatively be chosen as controllable companion characteristic form with $r \leq m$ given by 4.4.22 for which the characteristic polynomial can be written easily. The required

characteristic polynomial can thus be factored into r factors for $1 \leq r \leq m$. This gives rise to many alternative choices.

These alternative choices in the design procedure are illustrated by the following example.

Example 9.1 :

Given the system

$$\bar{A} = \begin{bmatrix} 0 & 0 & -3 \\ 2 & 0 & -7 \\ 0 & -1 & 0 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$$

with $|sI-\bar{A}| = s^3-7\,s - 6$

$$= (s+1)\ (s+2)\ (s-3)$$

Design a l.s.v.f. such that the closed loop poles are located at

$- 3,\ - 2 + j\ 2$

The required characteristic polynomial of the compensated system is

$$= (s+3)\ (s^2 + 4s + 8)$$
$$= s^3 + 7\ s^2 + 20\ s + 24$$

I) Let the compensated system have its evolution matrix A_{com} in the form

$$A_{com} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -20 & -7 \end{bmatrix}$$

Referring to example 4.1 conversion to different conical forms of (A, B) can be considered.

1) Following (A, B) equivalent to given ($\bar{A}$, $\bar{B}$) are in the mv controllable companion form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 6 & -1 & -6 \\ -2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$$

with $d_1 = 2$ and $d_2 = 1$.

Then the feedback matrix F should be such that

$$BF = A_{com} - A$$

i.e.
$$\begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -20 & -7 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 6 & -1 & -6 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -6 & 1 & 6 \\ -22 & -20 & -8 \end{bmatrix}$$

The unique solution of the above equations is :

$f_{11} = -50$, $f_{12} = -39$, $f_{13} = -10$

$f_{21} = -22$, $f_{22} = -20$, $f_{23} = -8$

2) Another equivalent pair in the mv controllable companion form of A and B is

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 1.5 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -0.5 & 1 \end{bmatrix}$$

Here also the only significant rows of either A or B are the m (=2), σ_k^{th} rows. Here $\sigma_1 = 1$, $\sigma_2 = 3$. Hence the above method of designing F can exactly be followed.

3) Third choice of equivalent pair (A, B) is :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 7 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -0.5 \\ 0 & -0.5 \\ 1 & -2.5 \end{bmatrix}$$

Here A is in the scalar controllable companion form, having only one significant row, viz the third one. This is to be modified to $\begin{bmatrix} -24 & -20 & -7 \end{bmatrix}$

Obviously the first row of F is

$$= \begin{bmatrix} -24 & -20 & -7 \end{bmatrix} - \begin{bmatrix} 6 & 7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -30 & -27 & -7 \end{bmatrix}$$

and second row, null.

i.e.

$$F = \begin{bmatrix} -30 & -27 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus rank of F = 1 = r, where $\sum_{i=1}^{r} d_i = n$ and $r < m$.

II) We can consider different forms for A_{com} also, having the same characteristic polynomial.

1) Let

$$A_{com} = A + BF = \left[\begin{array}{cc:c} 0 & 1 & 0 \\ -8 & -4 & 0 \\ 0 & 0 & -3 \end{array}\right]$$

We have to choose A, B in the mv controll. comp. form with $d_1 = 2$ and $d_2 = 1$. Thus

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 6 & -1 & -6 \\ -2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 1 \end{bmatrix}$$

can be chosen to obtain A_{com} of the above form with suitable design of F where 6 elements of F can be uniquely decided.

2) Another choice of A_{com} would be

$$A_{com} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & -4 \end{bmatrix}$$

It can be obtained for the equivalent pair (A, B) given by

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 1.5 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -0.5 & 1 \end{bmatrix}$$

having the same form i.e. same values of controllability indices d_1, d_2.

9.5 INCOMPLETE STATE FEEDBACK

The full state of the system is not always measurable. In such case it is desirable to have control over the placement of the closed loop poles by linear feedback from the measurable states only. If output vector only is measurable, then it is desirable that linear feedback from

$$y = Cx \qquad ...(9.5.1)$$

should be sufficient. Either of these feedbacks means a feedback from r measurable states or r linear combinations of states with $r < n$. Arbitrary allocation of all the poles is not always possible with such feedback. If an l.s.v.f. control

$$u = Fx \qquad ...(9.5.2)$$

can allocate all the poles in the desired locations, the control

$$\begin{aligned} u &= Ky \\ &= KCx \qquad ...(9.5.3) \end{aligned}$$

should be made equivalent to 9.5.2. Thus K is required to satisfy the relation

$$KC = F \qquad ...(9.5.4)$$

As C is not invertible, a unique K is not possible from 9.5.4 for given F. Note also that F for given arbitrary pole location is also not unique. If C_{RI} is a right inverse of C, multiplying both sides of 9.5.4 by C_{RI} gives

$$K = FC_{RI} \qquad ...(9.5.5)$$

However the K of 9.5.5 satisfies 9.5.4 only when

$$F\,C_{RI}\,C = F \qquad ...(9.5.6)$$

Thus the necessary and sufficient condition to have linear output feedback (l.o.f.) for arbitrary pole allocation is to get, by trial and error, a right inverse from many possible right inverses which satisfies 9.5.6 [M1']. Some sufficient conditions can be found in literature to allocate less than n poles arbitrarily [W2,K1]. It is, at present, a topic under development.

9.6 L.S.V.F. BY STATE ESTIMATION

If a system is observable, its n th order state can be completely determined by knowing the input and corresponding output if at least n independent observations are available. The linear state variable feedback needs continuous knowledge of the current state. This can be obtained by on line computer exactly by storing the necessary n observations. An alternative to on-line computation, which may be expensive and may introduce time delay in the system dynamics, is to estimate the state using analog computation which does not use storage or memory. Let the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Eu \end{aligned} \qquad \ldots(9.6.1)$$

be observable. We can write

$$\begin{aligned} \dot{x} + Ky &= Ax + Ky + Bu \\ \dot{x} &= (A + KC)\,x + (KE + B)\,u - Ky \end{aligned} \qquad \ldots(9.6.2)$$

If u and y can be measured, and initial conditions $x(o)$ are known, the above state equation can be solved for $x(t)$ by standard analog circuit using n integrators. Such a circuit

can be called as observer as it observes the state from output and input of the system. The block diagram of the system with observer will look as in Fig. 9.6.1.

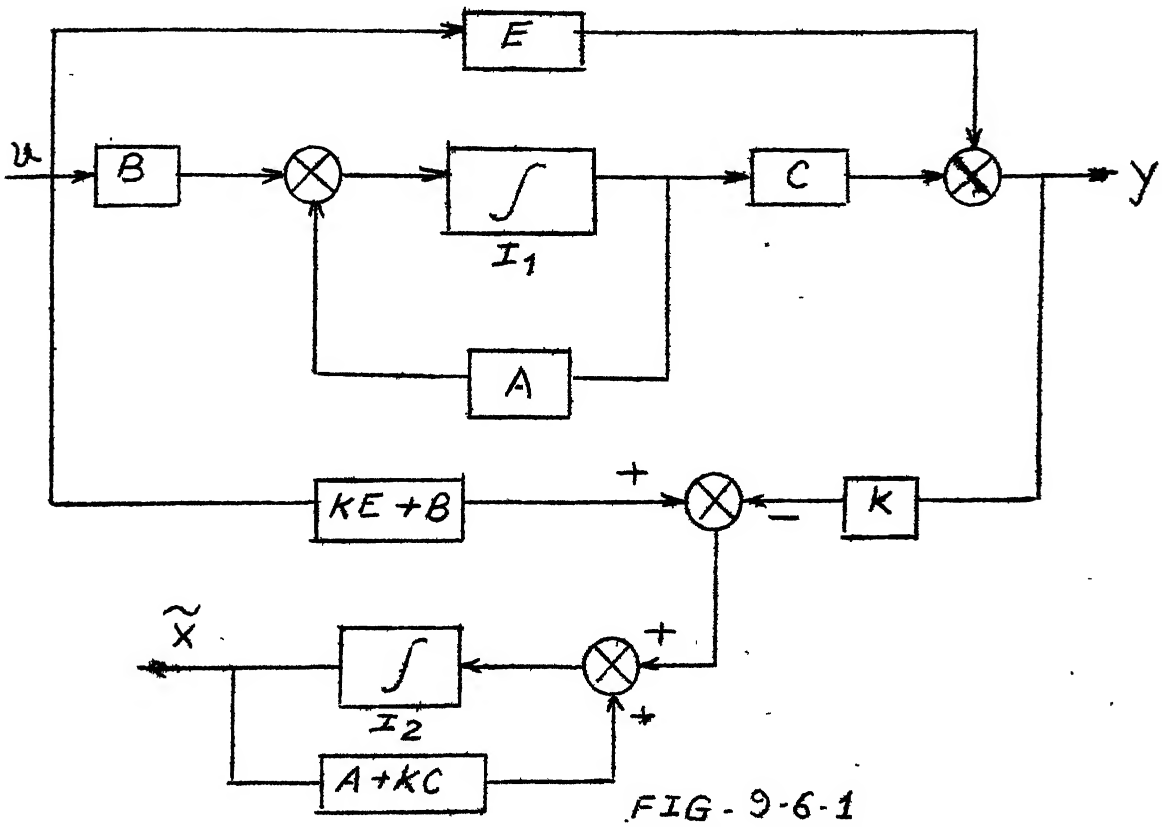

FIG - 9-6-1

The output of integrator block I_2 will be exactly equal to the system state if and only if the initial conditions given to I_2 are $x(o)$. But $x(o)$ is not known. Let then $\bar{x}$ be the output of I_2 for any arbitrary initial condition $\bar{x}(o)$.

Thus

$$\dot{\bar{x}} = (A + KC)\,\bar{x}(t) + (KE + B)\,u - Ky \qquad \ldots(9.6.3)$$

From 9.6.2 and 9.6.3

$$\dot{x} - \dot{\bar{x}} = (A + KC)\,(x - \bar{x}) \qquad \ldots(9.6.4)$$

Hence $(x - \bar{x}) = 0$ in the steady state.

i.e. In the steady state x and $\bar{x}$ are equal. The transient time during which x and $\bar{x}$ are not equal can be reduced and $\bar{x}$ can be made to reach quickly the steady state value equal to x for any arbitrary initial condition $\bar{x}(o)$ by designing K such that the roots of the characteristic polynomial of $[A + KC]$ give quickest possible response with sufficient stability.

Thus for good estimation of state, it is necessary to assign, arbitrarily, all the poles of the observer i.e. the eigenvalues of A + KC.

It is already proved that completely arbitrary eigenvalues can be assigned to A + BF by choosing proper F if (A, B) is a controllable pair. By dual procedure it can be proved that completely arbitrary eigenvalues can be assigned to A + KC if (A, C) is an observable pair. If (A, C) is unobservable, only observable eigenvalues of A can be modified by KC. Thus it is necessary that the unobservable modes be stable. The state can then be well estimated if the unobservable modes of A are enough stable and fast decaying.

Substituting for y from 9.6.1 in 9.6.3

$$\dot{\bar{x}} = (A + KC)\,\bar{x} + Bu - KCx \qquad \ldots(9.6.5)$$

Let us write

$$\bar{y} = I\,\bar{x} \qquad \ldots(9.6.6)$$

and then combine 9.6.1, 9.6.5 and 9.6.6 into a set of 2 n state equations and p + n output equations as,

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -KC & A+KC \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u \qquad \ldots(9.6.7)$$

and

$$\begin{bmatrix} y \\ \bar{y} \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} u \qquad \ldots(9.6.8)$$

If the system is now compensated through a constant gain matrix F, from the estimated state $\bar{x}$, then for the compensated system following equations can be written

$$\begin{aligned} u &= F\,\bar{x} + Gv \\ \dot{x} &= Ax + B\,F\bar{x} + B\,Gv \\ \dot{\bar{x}} &= (A + KC)\,\bar{x} + B\,F\bar{x} + BGv - KCx \\ y &= Cx + EF\bar{x} + EGv \\ \bar{y} &= \bar{x} \end{aligned} \qquad \ldots(9.6.9)$$

These 2 n state equations and corresponding p + n output equations of the compensated system can be written conveniently as

$$\begin{bmatrix} \dot{x} \\ \dot{x} - \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & A + KC \end{bmatrix} \begin{bmatrix} x \\ x - \bar{x} \end{bmatrix} + \begin{bmatrix} BG \\ 0 \end{bmatrix} v \qquad \ldots(9.6.10)$$

and

$$\begin{bmatrix} y \\ \bar{y} \end{bmatrix} = \begin{bmatrix} C + EF & -EF \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ x - \bar{x} \end{bmatrix} + \begin{bmatrix} EG \\ 0 \end{bmatrix} v \qquad \ldots(9.6.11)$$

From 9.6.10, the partial state $x - \bar{x}$ is not affected by V, and it is desirable, because we want to reduce $x - \bar{x}$ to zero quickly for any v. The closed loop poles of the combined 2 n ordered system are given by the roots of

$$\begin{vmatrix} sI - (A+BF) & BF \\ 0 & sI - A - KC \end{vmatrix} = 0 \qquad \ldots(9.6.12)$$

i.e. by

$$|sI-(A+BF)| = 0 \qquad \ldots(9.6.13)$$

and

$$|sI-A-KC| = 0 \qquad \ldots(9.6.14)$$

For arbitrary eigenvalue assignment for 9.6.13 and 9.6.14 it is necessary to have (A, B) controllable and (A, C) observable.

The observer discussed above is commonly known as Leunberger observer after its investigator.

Good design of such an observer requires proper determination of the observer poles. In order to make the settling time very small, it is necessary to have the observer gains sufficiently large. This tends to increase noise problems. So the observer design should be a good compromise between noise reduction and fast response. An optimal design of K for minimizing the effect of statistical noise is known as kalman filter.

9.7 REDUCED ORDER STATE OBSERVER

As the p outputs are available for measurement, p state variables can be directly obtained as shown below and an observer of order n-p only is sufficient to estimate remaining n-p state variables.

Let the given observable system be

$$\dot{x} = Ax + Bu$$

$$y = Cx + Eu \qquad \ldots(9.7.1)$$

where A and C are in the multivariable observable companion form, with observability indices d_j for $j = 1,2, \ldots, p$.

Thus A and C each have p significant columns $\sigma_k = \sum_{j=1}^{k} d_j$ for $k = 1,2,\ldots p$. Let us use the following notation in this section for deriving the state equations of the reduced order state observer.

$[x]_r$ → part of dimension r of the vector x

$[A_r]$ → A square matrix of dimension r obtained as a part of matrix A

$[A_{r,q}]$ → An r x q matrix, which is a part of matrix A

$$\therefore \quad y = Cx + Eu$$
$$= [C_p]\,[x]_p + Eu \qquad \ldots(9.7.2)$$

where C_p is p x p nonsingular matrix formed by p significant columns of C and $[x]_p$ is a p ordered vector with its elements equal to the state variables $x_{\sigma_1}, x_{\sigma_2} \ldots, x_{\sigma_p}$. These p state variables can be directly obtained from 9.7.2 as

$$[x]_p = [C_p]^{-1}\,[y - Eu] \qquad \ldots(9.7.3)$$

The remaining n-p state variables require observer for estimation. The state equations corresponding to these n-p state variables can be written as,

$$[\dot{x}]_{(n-p)} = [A_{(n-p),n}]\,x + [B_{(n-p),m}]u \qquad \ldots(9.7.4)$$

$$\therefore \quad [\dot{x}]_{(n-p)} = [A_{(n-p)}]\,[x]_{(n-p)} + [A_{(n-p),p}]\,[x]_p + [B_{(n-p),m}]u \qquad \ldots(9.7.5)$$

Substituting 9.7.3 in 9.7.5,

$$[\dot{x}]_{(n-p)} = [A_{(n-p)}]\ [x]_{(n-p)} + [A_{(n-p),p}]\ C_p^{-1}\ y$$

$$+ [B_{(n-p),\bar{m}}\ A_{(n-p),p}\ C_p^{-1}\ E]\ u \qquad \ldots(9.7.6)$$

This equation itself cannot be used for estimation of $[x]_{(n-p)}$ as the eigenvalues of $[A_{(n-p)}]$ do not have desirable location and are not adjustable also.

Hence an equivalent state may be defined as

$$[\bar{x}]_{(n-p)} = [x]_{(n-p)} - [P]\ [x]_p$$

where [P] is an (n-p)xp scalar matrix to be chosen later

Now, let the p equations

$$[\bar{x}]_p = [x]_p$$

be combined with 9.7.7(a) to form a set of n equations, 9.7.7(b) being the p, σ_kth, equations of this set. The resulting set of n algebraic equations can be written as

$$[\bar{x}]_n = T\ [x]_n \text{ say} \qquad \ldots(9.7.8)$$

where T is nonsingular and hence $[\bar{x}]_n$ and $[x]_n$ are equivalent.

Differentiating 9.7.7(a)

$$[\dot{\bar{x}}]_{(n-p)} = [\dot{x}]_{n-p} - [p]\ [\dot{x}]_p \qquad \ldots(9.7.9)$$

Substituting for $[\dot{x}]_{n-p}$ from 9.7.6,

$$[\dot{\bar{x}}]_{(n-p)} = [A_{(n-p)}]\ [x]_{n-p} + [A_{(n-p),p}]\ C_p^{-1}\ y$$

$$+ [B_{(n-p),\bar{m}}A_{(n-p),p}C_p^{-1}E]u - [p][\dot{x}]_p$$

$$\ldots(9.7.10)$$

$[\dot{x}]_p$ can be obtained as a part of given state equations 9.7.1 and can be written as

$$[\dot{x}]_p = [A_{p,n}]\,[x] + [B_{p,m}]u \qquad \ldots(9.7.11)$$

$$= [A_p]\,[x]_p + [B_{p,m}]u + [A_{p,(n-p)}][x]_{n-p} \qquad \ldots(9.7.12)$$

Substituting from 9.7.7(a) for $[x]_{(n-p)}$

$$[\dot{x}]_p = [A_p]\,[x]_p + [A_{p,(n-p)}]\,[\bar{x}]_{(n-p)} + [A_{p,(n-p)}]P[x]_p + [B_{p,m}]u \qquad \ldots(9.7.13)$$

Substituting $[\dot{x}]_p$ from 9.7.13 and $[x]_{n-p}$ from 9.7.7(a) into 9.7.10.

$$\begin{aligned}[\dot{\bar{x}}]_{n-p} = {} & [A_{(n-p)}][\bar{x}]_{(n-p)} + [A_{(n-p),p}]\,C_p^{-1}\,y \\ & + [A_{(n-p)}\,P]\,[x]_p + [B_{(n-p),m} - A_{(n-p),p}C_p^{-1}E]u \\ & - [P]\Big[[A_p]\,[x]_p + [A_{p,(n-p)}][\bar{x}]_{(n-p)} + \\ & + [A_{p,(n-p)}\,P]\,[x]_p + [B_{p,m}]u\Big] \end{aligned} \qquad \ldots(9.7.14)$$

Substituting for $[x]_p$ from 9.7.3 and rearranging

$$\begin{aligned}[\dot{\bar{x}}]_{n-p} = {} & \Big[[A_{(n-p)}] - [P\,A_{p,(n-p)}]\Big][\bar{x}]_{(n-p)} \\ & + \Big[[A_{(n-p),p}] + [A_{(n-p)}][P] - [P][A_p] \\ & - [P][A_{p,(n-p)}][P]\Big]\,C_p^{-1}\,y + [B_{(n-p)m} - P\,B_{p,m}]u \end{aligned}$$

$$+ \left[A_{(n-p),p} - A_{(n-p)} P + PA_p + PA_{p,(n-p)} P\right] C_p^{-1} Eu$$

$$= \left[\bar{A}_{(n-p)}\right] \left[\bar{x}\right]_{(n-p)} + My + \bar{B}u \text{ , say} \qquad \ldots(9.7.15)$$

Note that $\left[\bar{A}_{(n-p)}\right]$ of 9.7.15 happens to be a (n-p) square matrix in the multivariable observable companion form with its p, $(\sigma_k - k)$th significant columns for $k = 1,2,\ldots$ p. [with $\sigma_p - p = n-p$, the last column] equal to the p columns of $-\left[P\right]$ respectively.

Thus the desired pole location of the $(n-p)^{th}$ order observer can be directly translated into desired $\bar{A}$, assuming $\bar{A}$ either in the scalar observable companion form or in the multivariable observable companion form with its significant columns numbered $\sigma_k - k$ for $k = 1,2, \ldots$ p. Correspondingly the required values of elements of $\left[P\right]$ can be directly obtained. Then the state equations 9.7.15 of the observer can be calculated and realised. For arbitrary initial conditions, estimation of $\left[\bar{x}\right]_{(n-p)}$ can be correct only in the steady state. Hence the transient period should be reduced maintaining sufficient stability by proper allocation of poles of observer i.e. eigenvalues of $\left[\bar{A}\right]_{(n-p)}$.

<u>Example 9.2</u> : Consider the quadruple (A, B, C, E)

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 4 & 0 & -1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then $d_1 = 3$, $d_2 = 2$, $n = 5$, $m = 1$, $p = 2$

i) Let the desired $n-p = 3$ poles of observer be $-1, -2, -3$. Then the corresponding observable form of desired $[\bar{A}_{n-p}]$ is given by

$$[\bar{A}_3] = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

This requires P to be

$$P = \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix}$$

$$\therefore \; M = \Big[[A_{(n-p),p}] + [A_{(n-p)}] [P] - [P][A_p] - [P] [A_{p,(n-p)}] [P] \Big] \times C_p^{-1}$$

$$= \left[\begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} \right] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1}$$

$$\bar{B} = [B_{(n-p),m} - PB_{p,m}]$$

$$+ [A_{(n-p),p} - A_{(n-p)}P + P A_p + PA_{p,(n-p)}P] C_p^{-1}E$$

$$= \left[\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} +1 \\ 0 \end{bmatrix} \right]$$

$$+ \left[\begin{bmatrix} -2 & 0 \\ 4 & -1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix} + \right.$$

$$\left. \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 0 & 11 \\ -1 & 6 \end{bmatrix} \right] \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 95 \\ 159 \\ 61 \end{bmatrix}$$

Hence we can write the observer dynamics 9.7.15 and realise it.

ii) Let another choice of $[\bar{A}_{(n-p)}]$ in a different multivariable observable companion form be,

$$[\bar{A}_3] = \begin{bmatrix} 0 & -2 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

with its ch. poly

$= (s^2 + 3s + 2)\ (s + 3)$

$= (s + 1)\ (s + 2)\ (s + 3)$

as required

Then the desired P is given by

$$P = \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Hence M and $\bar{B}$, calculated as in the previous case happen to be

$$M = \begin{bmatrix} -14 & -6 \\ -13 & -10 \\ -3 & -7 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 21 \\ 32 \\ 3 \end{bmatrix}$$

Hence we get an alternative design of observer. Design of an observer of further reduced order can more conveniently be obtained using frequency domain techniques as explained in the next chapter.

Exercise 9 :

(9.1) Design linear state variable feedback to compensate the system of problem 4.2 to get the closed loop poles at $-1 \pm j\sqrt{2}$ and -2. Is the uncompensated system stable?

(9.2) Is the system of problem 4.3 stabilisable by l.s.v.f.? How many poles of the system can be arbitrarily located? Can they be placed in the desired positions by output feedback only?

(9.3) The triple (A, B, C) for a system is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Design an observer of order one to observe the state. Let the observer pole be at -5. Design the state feedback F to get the compensated system with poles at -1, -2, -3. Draw the block diagram of the compensated system. Find out the feedback transfer matrices and draw the system block diagram in the form of Fig. 10.2.1.

(9.4) What are the eigenvalues of $[A_{(n-p)}]$ of (9.7.6)? Why are they not adjustable?

(9.5) Verify that T of equation (9.7.8) is nonsingular.

(9.6) Find out the l.s.v.f. matrices F for the illustrative example (9.1-II).

(9.7) Convert the system of problem 9.3 into a single input controllable system and repeat the problem. Can the system state be observed from a single output?

+ + +

CHAPTER - 10

FREQUENCY DOMAIN DESIGN

10.1 INTRODUCTION

The linear state variable feedback can be used to arbitrarily locate the controllable poles of the system. If the full state is not available for feedback, it can be obtained using an observer. The observer requires its inputs, from the control input and output of the system. Such a control scheme shown in Fig. 9.1 can be looked upon as a system compensated by a dynamic compensator with the compensator inputs coming from y and u in Fig. 9.1. For this situation it is more convenient to use the transfer matrix description of the system and frequency domain techniques for analysis and design. Hence in this chapter we will study a frequency domain configuration similar to Fig. 9.1 and its utility for obtaining arbitrary pole placement. It is observed that the frequency domain configuration is of more general use than only arbitrary pole allocation. L.s.v.f. for pole allocation will be considered as a special case obtained from the general dynamic feedback compensation in the frequency domain.

The other major application of the configuration is achieving noninteractive control. It is discussed later in this chapter. By noniteration we mean here the conversion of the given system matrix $T(s)$ into a diagonal matrix $T_d(s)$ by

compensation. This means that any system output is affected by only one system input. This is also referred to as dynamic decoupling. Static decoupling is conversion of the transfer matrix to diagonal form during the steady state only.

Conditions for diagonalisation by linear output feedback (l.o.f.) can be derived by considering l.o.f. as a special case of general compensation obtained from the dynamic feedback scheme.

10.2 ARBITRARY POLE ALLOCATION

Let the p x m system transfer matrix

$$T(s) = R(s)\ P^{-1}(s) \qquad \ldots(10.2.1)$$

be a proper transfer matrix with R(s) and P(s) relatively right-prime and P(s) column proper. Such a decomposition of given T(s) is always possible as shown previously in sections 7.4 and 7.6. The advantages of this decomposition are :

i) The equivalent differential operator representation is of minimal order and hence, for the column proper P(s),

ii) the sum of column degrees = d $|P(s)|$ is smallest possible. Resulting smallest possible column degrees are desirable to reduce the dimension of the matrix $S_e(s)$ defined later in the design algorithm and to simplify computation. Then, for closed loop system configuration shown in Fig. 10.2.1

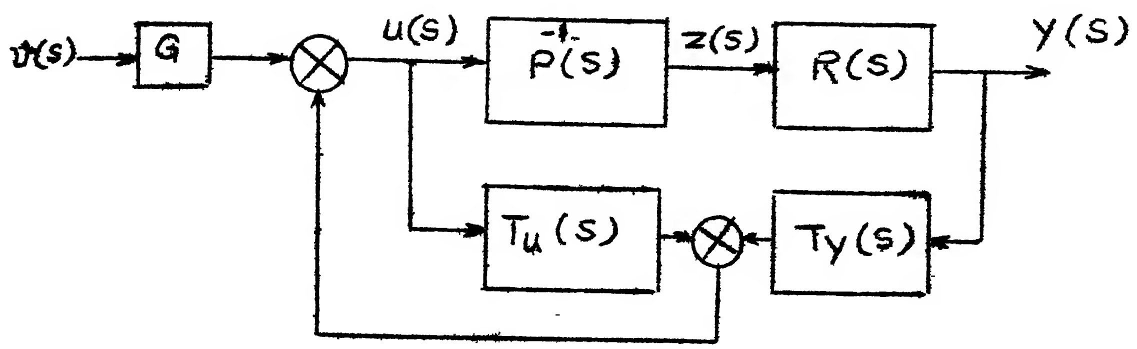

FIG. 10.2.1

$$u(s) = G\ v(s) + T_u(s)\ P(s)\ z(s) + T_y(s)\ R(s)\ z(s) = P(s)\ z(s) \qquad \ldots(10.2.2)$$

Let us design proper transfer matrices $T_u(s)$ and $T_y(s)$, in the compensator such that,

$$T_u(s)\ P(s) + T_y(s)\ R(s) \triangleq F(s) \qquad \ldots(10.2.3)$$

is a polynomial matrix. The type of closed loop transfer matrix obtained with such feedback transfer matrices can be derived as follows. Let

$$P_F(s) = P(s) - F(s) \qquad \ldots(10.2.4)$$

Then from (10.2.2), (10.2.3) and (10.2.4)

$$G\ v(s) = P_F(s)\ z(s)$$

$$\therefore\ z(s) = P_F^{-1}(s)\ G\ v(s)$$

and $$y(s) = R(s)\ z(s) = R(s)\ P_F^{-1}(s)\ G\ v(s) \qquad \ldots(10.2.5)$$

Thus the closed loops transfer matrix

$$T_{cl}(s) = R(s)\ P_F^{-1}(s)\ G$$

$$= R(s)\ P^{-1}(s)\ P(s)\ P_F^{-1}(s)\ G$$

$$= T(s)\ T_c(s)\ G$$

with $\quad T_c(s) = P(s)\ P_F^{-1}(s)$, ...(10.2.6)

representing an open loop precompensation equivalent to the dynamic feedback compensator.

If now the design objective is to allocate the closed loop poles, design specification must be given by specifying $P_F(s)$.

G is a constant matrix and can be different than unit matrix only when the input vector needs any modification as explained later.

The controllable and observable poles of the closed loop system are the roots of $|P_F(s)|$. Hence the given configuration is ideal for locating these in the desired positions. $P_F(s)$ should be chosen such that $|P_F(s)|$ is a stable polynomial with its roots equal to the desired closed loop poles. It is desirable to have $T_{cl}(s)$ in the form of a proper transfer matrix. Hence $P_F(s)$ can be restricted to be column proper and to satisfy the condition that any column degree of $P_F(s)$ should be larger than or equal to the corresponding column degree of $R(s)$.

i.e. $\quad d_c\ P_F(s) \geq d_c\ R(s)$ ...(10.2.7)

The polynomial matrix $F(s)$ obtained from the specified $P_F(s)$, using (10.2.5), has to satisfy the condition,

$$d_c\, F(s) \leq d_c\, P(s) \qquad \ldots(10.2.8)$$

as explained in the design algorithm.

In order to satisfy (10.2.8), referring to equation (10.2.4), it is necessary to choose $P_F(s)$ such that,

$$d_c\, P_F(s) \leq d_c\, P(s) \qquad \ldots(10.2.9)$$

Thus for arbitrary pole placement the matrix $P_F(s)$ can be specified with any desirable stable poles, with the restriction

$$d_c\, R(s) \leq d_c\, P_F(s) \leq d_c\, P(s) \qquad \ldots(10.2.10)$$

as obtained by combining (10.2.7) and (10.2.8). It may be noted that this general frequency domain compensation for arbitrary pole allocation allows

$$d\, |P_F(s)| \leq d\, |P(s)| \qquad \ldots(10.2.11)$$

That is, it allows the minimal order of the corresponding closed loop state space representation to be smaller than that for the original system. 10.2.11 may result in non proper transfer matrix $T_c(s)$ of 10.2.6. However this should not matter as far as the transfer matrices $T_u(s)$ and $T_y(s)$ to be realised are proper transfer matrices.

Let q(s) be the l.c.m. of the denominators of $T_u(s)$ and $T_y(s)$. These transfer matrices can then be written as

$$T_u(s) = \frac{K(s)}{q(s)}$$

and $$T_y(s) = \frac{H(s)}{q(s)} \qquad \ldots(10.2.12)$$

Then (10.2.3) can be written as

$$q(s)\ F(s) = K(s)\ P(s) + H(s)\ R(s) \qquad \ldots(10.2.13)$$

Thus the problem of designing a closed loop compensated system can be stated as :

For the specified $P_F(s)$, design the polynomial matrices $K(s)$ and $H(s)$ to satisfy (10.2.13) for an arbitrarily chosen, lowest possible order, stable polynomial $q(s)$ such as to have $T_u(s)$ and $T_y(s)$ of 10.2.12 in the form of proper transfer matrices, in order to make them realisable.

Choosing smallest possible order of $q(s)$ helps in reducing the order of the compensation. If $q(s)$ is of degree g, then the order of the minimal realisation of the compensator transfer matrices $T_u(s)$ and $T_y(s)$ is less than or equal to gm.

The algorithm for solving this design problem is given in the following section.

10.3 Design Algorithm : Dynamic Feedback Compensation

Let the highest degree element of $K(s)$ or $H(s)$ be of degree g. Then in order to make $T_u(s)$ and $T_y(s)$ proper, with smallest degree of $q(s)$, let,

$$q(s) = q_0 + q_1\ s + q_2\ s^2 + \ldots + q_g\ s^g$$

$$K(s) = K_0 + K_1\ s + K_2\ s^2 + \ldots + K_g\ s^g$$

and

$$H(s) = H_0 + H_1\ s + H_2\ s^2 + \ldots + H_g\ s^g \qquad \ldots(10.3.1)$$

where q_o, q_1 ... q_g are scalar coefficients and K_i, H_i for $i = 0, 1, 2 \ldots g$ are scalar matrix coefficients. Then 10.2.13 can be written as,

$$\left[q_o I_m \; q_1 I_m \ldots q_g I_m \right] \begin{bmatrix} F(s) \\ sF(s) \\ \cdot \\ \cdot \\ \cdot \\ s^g F(s) \end{bmatrix} = \left[K_o, K_1 \ldots K_g \right] \begin{bmatrix} P(s) \\ sP(s) \\ \cdot \\ \cdot \\ \cdot \\ s^g P(s) \end{bmatrix}$$

$$+ \left[H_o, H_1 \ldots H_g \right] \begin{bmatrix} R(s) \\ sR(s) \\ \cdot \\ \cdot \\ \cdot \\ s^g R(s) \end{bmatrix} \qquad \ldots(10.3.2)$$

Let us define

$$S_g(s) = \begin{bmatrix} 1 & 0 & \cdot & 0 \\ s & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s^{d_1+g} & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot \\ \cdot & s & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & s^{d_2+g} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & s^{d_m+g} \end{bmatrix} \qquad \ldots(10.3.3)$$

The dimensions of $S_g(s)$ are k x m, with

$$k = \sum_{j=1}^{m} d_j + m(g+1)$$
$$= n + m(g+1) \qquad \ldots(10.3.4)$$

where $n = \sum_{j=1}^{m} d_j \qquad \ldots(10.3.5)$

is the degree of $|P(s)|$ as, $d_1\ d_2 \ldots d_m$ are the column degrees of $P(s)$ and $P(s)$ is column proper. Let the polynomial matrices in 10.3.2 be written as,

$$\begin{bmatrix} P(s) \\ sP(s) \\ \cdot \\ \cdot \\ s^g P(s) \end{bmatrix} = \alpha_g\, S_g(s) \qquad \ldots(10.3.6)$$

$$\begin{bmatrix} R(s) \\ sR(s) \\ \cdot \\ \cdot \\ s^g R(s) \end{bmatrix} = \beta_g\, S_g(s) \qquad \ldots(10.3.7)$$

and

$$\begin{bmatrix} F(s) \\ sF(s) \\ \cdot \\ \cdot \\ s^g F(s) \end{bmatrix} = \gamma_g\, S_g(s) \qquad \ldots(10.3.8)$$

where α_g, β_g, γ_g are scalar matrices. Then 10.3.2 can be written in a compact notation as,

$$[\overline{Q}_g]\ \gamma_g\, S_g(s) = [\overline{K}_g]\ \alpha_g\, S_g(s) + [\overline{H}_g]\ \beta_g\, S_g(s) \qquad \ldots(10.3.9)$$

where

$$[\bar{Q}_g] = [q_0 I_m \; q_1 I_m \; q_2 I_m \cdots q_g I_m] \qquad \ldots(10.3.10)$$

$$[\bar{K}_g] = [K_0 \; K_1 \ldots K_g] \qquad \ldots(10.3.11)$$

and

$$[\bar{H}_g] = [H_0 \; H_1 \ldots H_g] \qquad \ldots(10.3.12)$$

Note that 10.3.9 is a polynomial matrix equation. Corresponding elements of both sides of equations must be equal. This requires that in each pair of corresponding elements, the coefficients of same power of s in the corresponding elements must be equal. Thus the polynomial matrix equation 10.3.9 can be replaced by the scalar matrix equation

$$[\bar{Q}_g] \; \gamma_g = [\bar{K}_g] \; \alpha_g + [\bar{H}_g] \; \beta_g \qquad \ldots(10.3.13)$$

The design problem now consists of solving 10.3.13. That is we have to obtain $[\bar{K}_g]$, $[\bar{H}_g]$ for $[\bar{Q}_g]$ corresponding to arbitrarily assumed stable q(s) of smallest possible order for which solution of 10.3.13 exists.

Let us examine the equation 10.3.13 for smallest value of g i.e. for g = 0. Then 10.3.13 becomes

$$[\bar{Q}_0] \; [\gamma_0] = [\bar{K}_0] \; \alpha_0 + [\bar{H}_0] \; \beta_0 \qquad \ldots(10.3.14)$$

Note that

$$[\bar{Q}_0] = I_m \text{ for a monic polynomial } q(s)$$

$$[\bar{K}_0] = [K_0] \text{ and } [\bar{H}_0] = [H_0] \qquad \ldots(10.3.15)$$

So, 10.3.14 can be written as

$$\gamma_o = K_o\ \alpha_o + H_o\ \beta_o \qquad \ldots(10.3.16)$$

Equation 10.3.16 can be solved for K_o and H_o, if and only if rows of γ_o are linearly dependent on the rows of α_o and β_o i.e. if and only if

$$\rho \begin{bmatrix} \alpha_o \\ \cdots \\ \beta_o \end{bmatrix} = \rho \begin{bmatrix} \alpha_o \\ \beta_o \\ \gamma_o \end{bmatrix} . \qquad \ldots(10.3.17)$$

Thus 10.3.17 becomes a necessary and sufficient condition to have a zero order compensation for the specified $P_F(s)$ as γ_o depends on the specified $P_F(s)$. Moreover, if

$$\rho\, [\beta_o] = \rho \begin{bmatrix} \beta_o \\ \gamma_o \end{bmatrix} \qquad \ldots(10.3.18)$$

K_o can be assumed equal to zero and an l.o.f. (linear output feedback) compensation can be obtained for the specified $P_F(s)$.

If however none of 10.3.18 or 10.3.17 is satisfied, we have to try to solve 10.3.13 for $g = 1$.
Equation 10.3.13 now becomes,

$$[\bar{Q}_1]\ \gamma_1 = [\bar{K}_1]\ \alpha_1 + [\bar{H}_1]\ \beta_1 \qquad \ldots(10.3.19)$$

It may be noted that the matrices γ_1, α_1, β_1 can be obtained directly from γ_o, α_o, and β_o respectively. Equation 10.3.19 has a solution if and only if rows of $\bar{Q}_1\ \gamma_1$ are linearly dependent on the rows of α_1 and β_1 i.e. if and only if,

$$\rho\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix} = \rho\begin{bmatrix}\alpha_1\\ \beta_1\\ \bar{Q}_1\ \gamma_1\end{bmatrix} \qquad \text{...(10.3.20)}$$

Note that this condition is dependent on the choice of $\bar{Q}_1$ i.e. the choice of q_1 as $q_o = 1$. Thus for the specified $P_F(s)$ and specified q_1, the compensation of order smaller than or equal to m is possible if and only if 10.3.20 is satisfied.

10.4 ORDER OF THE COMPENSATOR

In general, for a specified $P_F(s)$ and specified polynomial q(s) of degree g, a compensator of order less than or equal to mg is possible if and only if

$$\rho\begin{bmatrix}\alpha_g\\ \beta_g\end{bmatrix} = \rho\begin{bmatrix}\alpha_g\\ \beta_g\\ \bar{Q}_g\ \gamma_g\end{bmatrix} \qquad \text{...(10.4.1)}$$

Equation 10.4.1 is certainly satisfied independent of the choice of $\bar{Q}_g$, for the smallest value of g for which

(a) $\begin{bmatrix}\alpha_g\\ \beta_g\end{bmatrix}$ has number of rows equal to or larger than the number of columns and

(b) $\begin{bmatrix}\alpha_g\\ \beta_g\end{bmatrix}$ has full rank. ...(10.4.2)

However 10.4.1 may be satisfied for a smaller value of g also. As it is desirable to find out the smallest value of g for which 10.4.1 is satisfied it is necessary to apply the test 10.4.1 for all values of g starting from zero. But the test needs to be continued, at the most, till the

As the compensator order is directly related to m, the number of inputs, it is advisable to represent the given system by an equivalent controllable system using the smallest number of inputs as explained in Section 4.4.2. In that case the scalar matrix G can be used to convert the m-input vector to an r-input vector, with $r \leq m$.

A dual algorithm can be formed to get compensators of order less than or equal to $p(\mu-1)$, where μ is the controllability index of the system. In this algorithm the transfer matrix $T(s)$ would be represented in the form $\bar{P}^{-1}(s)\,\bar{Q}(s)$ where $\bar{P}(s)$ and $\bar{Q}(s)$ are relatively left prime and $\bar{P}(s)$ is row proper.

10.5 EQUIVALENCE TO L.S.V.F.

Let the S.S. quadruple (A, B, C, E) represent the uncompensated system, so that

$$\begin{aligned} T(s) &= C\,(sI-A)^{-1}\,B + E \\ &= R(s)\,P^{-1}(s) \end{aligned} \qquad \ldots(10.5.1)$$

Then referring to (9.2.6) and (9.2.9) the closed loop transfer matrix obtained by l.s.v.f. is

$$T_{cl}(s) = R(s)\,P_F^{-1}(s)\,G \qquad \ldots(10.5.2)$$

Referring to (9.2.10), (7.4.11) and (7.4.6) the matrix F(s) corresponding to the $P_F(s)$ of (10.5.2) has to satisfy the condition,

$$d_c\,F(s) < d_c\,P(s) \qquad \ldots(10.5.3)$$

Thus if the complete state is available for feedback, the additional restrictions on the choice of the column proper $P_F(s)$, specified corresponding to a desired pole allocation, are

i) $d_c\ P_F(s) = d_c\ P(s)$

and

ii) $\gamma_c\ [G\ P_F(s)] = \gamma_c\ [P(s)]$ ...(10.5.4)

where G is a constant invertible matrix.

The dyhamic feedback compensator can then generate a transfer matrix $R(s)\ [G\ P_F(s)]^{-1}$ which, when post multiplied by a forward constant matrix precompensator G, gives $R(s)\ P_F^{-1}(s)$ as the closed loop transfer matrix with desired pole location.

$$F(s) = G\ P_F(s) - P(s) \qquad \text{...(10.5.5)}$$

The equivalent l.s.v.f. matrix F can be obtained using (9.2.10). Thus if the complete state is available for feedback the zero order compensator transfer matrix is F. Otherwise F(s) of (10.5.5) can be used for designing the dynamic compensator using the algorithm of Section 10.4.

Following are two numerical examples illustrating the design of l.o.f. and a low order compensator by the above algorithm.

Example 10.1 :

Let $T(s) = R(s)\ P^{-1}(s)$

$$= \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ -1 & s-1 \end{bmatrix}^{-1}$$

with R(s) and P(s) relatively right prime.

Let

$$P_F(s) = \begin{bmatrix} s^2+s+1 & 0 \\ 1 & s+1 \end{bmatrix}$$

giving closed loop transfer matrix $T_{cl}(s)$ as

$$T_{cl}(s) = \begin{bmatrix} \dfrac{s+1}{s^2+s+1} & 0 \\ \dfrac{s}{(s+1)\,(s^2+s+1)} & \dfrac{1}{s+1} \end{bmatrix}$$

Then,

$$S_o(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}$$

and $F(s) = P(s) - P_F(s)$

$$= \begin{bmatrix} -s-1 & 0 \\ -2 & -2 \end{bmatrix}$$

Therefore,

$$\alpha_o = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\beta_o = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\gamma_o = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 0 \end{bmatrix}$$

It is easy to verify that,

$$\rho \begin{bmatrix} \alpha_o \\ \beta_o \end{bmatrix} = \rho \begin{bmatrix} \alpha_o \\ \beta_o \\ \gamma_o \end{bmatrix} = 4$$

Thus zero order compensation is possible. Moreover,

$$\rho \begin{bmatrix} \beta_o \end{bmatrix} = \rho \begin{bmatrix} \beta_o \\ \gamma_o \end{bmatrix} = 2$$

Hence the desired compensation is possible using l.o.f. only with $\begin{bmatrix} K_o \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$.

Equation 10.4.4 can then be written as,

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} H_o^T = \begin{bmatrix} -1 & -2 \\ -1 & 0 \\ 0 & 0 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

Considering any two linearly independent equations of the above 5 equations and choosing 2nd and 4th for convenience,

$$H_o^T = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = H_o$$

is the l.o.f. matrix giving the desired $T_{cl}(s)$.

Example 10.2 :

Let R(s) and P(s) of the system be as in Example 10.1. But let the specified $P_F(s)$ be

$$P_F(s) = \begin{bmatrix} s^2+2s+2 & 0 \\ s+1 & s+1 \end{bmatrix}$$

corresponding to a pole specification : closed loop poles to be located at -1, -1 $\pm$ j. Corresponding desired closed lopp transfer matrix is

$$T_{cl}(s) = \begin{bmatrix} \frac{s+1}{s^2+2s+2} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

Then,

$$F(s) = \begin{bmatrix} -2s-2 & 0 \\ -s-2 & -2 \end{bmatrix}$$

$S_e(s)$, α_o and β_o the same as in example 1

and

$$\gamma_o = \begin{bmatrix} -2 & -2 & 0 & 0 & 0 \\ -2 & -1 & 0 & -2 & 0 \end{bmatrix}$$

In this case

$$\rho \begin{bmatrix} \alpha_o \\ \beta_o \end{bmatrix} \neq \rho \begin{bmatrix} \alpha_o \\ \beta_o \\ \gamma_o \end{bmatrix}$$

Hence zero order compensator is not possible. Then let $g = 1$. Then,

$$\alpha_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Note that in the first 2 (= m) rows of α_1, columns 1, 2, 3 and 5, 6 are equal to columns 1 to 5 of α_o. The 4th and 7th columns of the first two rows are zero. The next two (= m) rows of α_1 are obtained by shifting the columns of first two rows each by 1 place to the right. The last column of the first two rows gets shifted to 1st column of the next two rows. β_1 and γ_1 can also be obtained mechanically from β_o and γ_o using above procedure, instead of calculating them using 10.3.6, 10.3.7 and 10.3.8.

$$\beta_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\gamma_1 = \begin{bmatrix} -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & -2 & 0 \end{bmatrix}$$

Then

$$\rho \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = 7, \text{ the full rank}$$

Hence

$$\rho \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \overline{Q}_1 \quad \gamma_1 \end{bmatrix} = 7$$

for any choice of q(s) of first order.

$\therefore$ Let the stable q(s) be,

$$q(s) = s+2$$

Then

$$\overline{Q}_1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

and hence

$$\begin{bmatrix} \alpha_1^T & \beta_1^T \end{bmatrix} \begin{bmatrix} \overline{K}_1^T \\ \overline{H}_1^T \end{bmatrix} = \begin{bmatrix} \gamma_1 \end{bmatrix}^T \begin{bmatrix} \overline{Q}_1^T \end{bmatrix}$$

gives

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{K}_1^T \\ \bar{H}_1^T \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -6 & -4 \\ -2 & -1 \\ 0 & 0 \\ 0 & -4 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

The solution for $\bar{K}_1$, $\bar{H}_1$ is not unique as

$$\rho \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \rho \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \bar{Q}_1 \quad \gamma_1 \end{bmatrix} = 7 < (r+1)\ (m+p),$$

which is equal to 8. It can be verified that one possible solution gives

$$K_o = \begin{bmatrix} 0 & 0 \\ +3 & -2 \end{bmatrix} \qquad H_o = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad H_1 = \begin{bmatrix} -2 & 0 \\ -4 & 0 \end{bmatrix}$$

Therefore

$$T_y(s) = -\begin{bmatrix} 2 & 0 \\ \frac{4s}{s+2} & \frac{6}{s+2} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 4s & 6 \end{bmatrix}$$

and

$$T_u(s) = -\begin{bmatrix} 0 & 0 \\ \frac{3}{s+2} & \frac{2}{s+2} \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$$

Thus the order of the compensator is 1, which is less than either $m(\gamma-1)$ or $p(\mu-1)$.

10.6 DYNAMIC DECOUPLING : INVERTIBLE T(s)

Let us now show that the dynamic feedback compensation scheme of Fig. 10.1 is more versatile than only being useful for arbitrary pole allocation. Another major goal of compensation, the same configuration can achieve, is the dynamic decoupling.

In this case the equivalent open loop compensation transfer matrix $T_c(s)$ of (10.2.6) is required to convert the given system into a number of noninteracting single input single output systems. This requires conversion of the p x m system transfer matrix T(s) into a p x p diagonal matrix $T_d(s)$ by the m x p precompensator $T_c(s)$. i.e.,

$$T_d(s) = T(s)\, T_c(s) \qquad \ldots(10.6.1)$$

As $T_d(s)$ is a diagonal matrix and none of its diagonal elements is desired to be zero, $T_d(s)$ is nonsingular. Thus it is necessary that rank of each of T(s) and $T_c(s)$ be atleast equal to p. This is equivalent to requiring

$$p \leq m \qquad \ldots(10.6.2)$$

as a necessary condition for diagonalisation.

Let us first consider in this section, the case p = m. That is, we consider an invertible p x p transfer matrix T(s). For an invertible T(s) we can write (10.6.1) as

$$T_c(s) = T^{-1}(s)\, T_d(s)$$

$$= P(s)\, R^{-1}(s)\, T_d(s)$$

i.e. $P^{-1}(s)\ T_c(s) = R^{-1}(s)\ T_d(s)$ ...(10.6.3)

Let us now examine the possibility of using the dynamic feedback compensation of Fig. 10.2 for satisfying (10.6.3). The compensation scheme can give [Refer 10.2.6]

$$P^{-1}(s)\ T_c(s) = P_F^{-1}(s) \qquad \text{...(10.6.4)}$$

Thus diagonalisation can be done by this scheme if we choose $T_d(s)$ such that

$$R^{-1}(s)\ T_d(s) = P_F^{-1}(s) \qquad \text{...(10.6.5)}$$

The obvious choice for such a $T_d(s)$ is,

$$T_d(s) = P_d^{-1}(s) \qquad \text{...(10.6.6)}$$

giving

$$P_F(s) = P_d(s)\ R(s) \qquad \text{...(10.6.7)}$$

If we can separate a diagonal polynomial matrix factor $R_d(s)$ from $R(s)$ as

$$R(s) = R_d(s)\ \bar{R}(s) \qquad \text{...(10.6.8)}$$

then $T_d(s)$ can be chosen as

$$T_d(s) = R_d(s)\ P_d^{-1}(s) \qquad \text{...(10.6.9)}$$

Then from (10.6.5), (10.6.8) and (10.6.9)

$$P_F(s) = P_d(s)\ \bar{R}(s) \qquad \text{...(10.6.10)}$$

In both (10.6.7) and (10.6.10), $P_d(s)$ is to be chosen with

stable diagonal elements such that, $T_d(s)$ is a proper transfer matrix and $P_F(s)$ satisfies (10.2.9) i.e.

$$d_c\ P_F(s) \leq d_c\ P(s) \qquad \ldots(10.6.11)$$

It can be proved that such a choice of $P_d(s)$ is possible for all proper transfer matrices $R(s)\ P^{-1}(s)$. The proof of this fact is given below.

For satisfying (10.6.11), it is sufficient to choose $P_d(s)$ such that $P_F(s)\ P^{-1}(s)$ is proper. But,

$$\begin{aligned} P_F(s)\ P^{-1}(s) &= P_d(s)\ \bar{R}(s)\ P^{-1}(s) \\ &= P_d(s)\ R_d^{-1}(s)\ R_d(s)\ \bar{R}(s)\ P^{-1}(s) \\ &= P_d(s)\ R_d^{-1}(s)\ T(s) \qquad \ldots(10.6.12) \end{aligned}$$

For each side of (10.6.12) and $T_d(s)$ to be proper transfer matrix it is sufficient to have for all $i = 1, 2, \ldots p$,
$0 \leq$ Degree of i^{th} row of $P_d(s)$ - Degree of i^{th} row of $R_d(s) \leq$ minimum of [degree of denominator of j^{th} element of i^{th} row of $T(s)$ - degree of numerator of j^{th} element of i^{th} row of $T(s)$]
for $j = 1,2,\ldots, p$ (10.6.13)

As $T(s)$ is a proper transfer matrix r.h.s. of (10.6.13) is equal to or greater than zero. Hence $P_d(s)$ can always be chosen to satisfy (10.6.13).

Thus the dynamic feedback configuration of Fig. 10.2.1 can always be used for decoupling. The closed loop transfer matrix is $T_d(s)$ where all the poles, namely the zeros of

$|P_d(s)|$ can be chosen arbitrarily. The poles of compensator are the zeros of q(s) which can be made stable by arbitrary choice, as explained in the design algorithm of dynamic feedback compensation.

Example 10.3 :

Let

$$T(s) = \begin{bmatrix} \frac{s+1}{s^2} & \frac{1}{s} \\ \frac{1}{s^2} & \frac{1}{s^2} \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & s \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix}^{-1}$$

Desired closed loop transfer matrix is diagonal matrix $T_d(s)$.

Let $T_d(s) = R_d(s)\, P_d^{-1}(s)$

where $R_d(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

formed by the greatest common factors of rows of R(s).

Then $\bar{R}(s) = R_d^{-1}(s)\, R(s)$

$$= \begin{bmatrix} s+1 & s \\ 1 & 1 \end{bmatrix}$$

Then according to (10.6.13) 1^{st} and 2^{nd} diagonal elements of $P_d(s)$ should have degrees ≤ 1 and 2 respectively. Let us have

$$P_d(s) = \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}$$

so that the closed loop matrix

$$R_d(s)\ P_d^{-1}(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

Then as per (10.6.10)

$$P_F(s) = P_d(s)\ \bar{R}(s)$$

$$\begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} s+1 & s \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (s+1)^2 & s(s+1) \\ (s+2) & (s+2) \end{bmatrix}$$

Design of a dynamic feedback compensation for getting the above $P_F(s)$ is already done in example 10.1.

10.7 DYNAMIC DECOUPLING BY L.S.V.F.

If diagonalisation of transfer matrix is to be done with l.s.v.f. only, the choice of $P_d(s)$ and hence of $P_F(s)$ through (10.6.10) must be such as to have $R_d(s)\ P_d^{-1}(s)$ a proper transfer matrix and to satisfy conditions (10.5.4). Choice of a $P_d(s)$ satisfying these conditions is not possible for all proper transfer matrices. As proved below, for the choice of $P_d(s)$ satisfying (10.5.4) to be possible, it is necessary and sufficient to have

$$B^{*}\left[T(s)\right] = \lim_{s \to \infty}\left[D(s)\ T(s)\right]$$

$$= \text{a nonsingular constant matrix} \qquad \ldots(10.7.1)$$

$D(s)$ is defined as the diagonal matrix of i^{th} element s^{k_i}, where K_i is the minimum of [the degree of denominator – degree of numerator] for all the elements of i^{th} row of $T(s)$. Proof of sufficiency and necessity follows :

Sufficiency : Referring to (10.6.12)

$$\lim_{s \to \infty} [P_F(s)\ P^{-1}(s)]$$

$$= \lim_{s \to \infty} [P_d(s)\ R_d^{-1}(s)\ T(s)] \qquad \ldots(10.7.2)$$

If we choose the diagonal matrix $P_d(s)$ with each element stable and with degree of i^{th} element $= K_i$ + degree of i^{th} element of $R_d(s)$ then, from (10.7.1).

$$\lim_{s \to \infty} [P_d(s)\ R_d^{-1}(s)\ T(s)] \qquad \ldots(10.7.3)$$

$$= \lim_{s \to \infty} [D(s)\ T(s)] \qquad \ldots(10.7.3)$$

Then from (10.7.1), (10.7.2) and (10.7.3)

$$B^*[T(s)] = \lim_{s \to \infty} [P_F(s)\ P^{-1}(s)] \qquad \ldots(10.7.4)$$

Let (10.7.4) be a constant nonsingular matrix denoted by E. $P_F(s)\ P^{-1}(s)$ can be expressed as

$$P_F(s)\ P^{-1}(s) = E + R_1(s)\ P^{-1}(s) \qquad \ldots(10.7.5)$$

where $R_1(s)\ P^{-1}(s)$ is strictly proper part of $P_F(s)\ P^{-1}(s)$ and E is a constant matrix.(10.7.5) can be written as

$$P_F(s) = E\ P(s) + R_1(s) \qquad \ldots(10.7.6)$$

where

$$d_c\ R_1(s) < d_c\ P(s) \qquad \ldots(10.7.7)$$

Hence

$$d_c\ P_F(s) = d_c\ P(s) \qquad \text{...(10.7.8)}$$

Moreover

$$\gamma_c\ [E^{-1}P_F(s)] = \gamma_c\ P(s) \qquad \text{...(10.7.9)}$$

Note that (10.7.8) and (10.7.9) are the same as (10.5.4) with $G = E^{-1}$. Hence (10.7.1) is a sufficient condition for diagonalisation by l.s.v.f.

<u>Necessity</u> : If diagonalisation is obtained by l.s.v.f., we have $P_F(s)$ such that

$$d_c\ P_F(s) = d_c\ P(s)$$

and $$\gamma_c\ [G\ P_F(s)] = \gamma_c\ [P(s)] \qquad \text{...(10.7.10)}$$

and $R(s)\ P_F^{-1}(s)\ G^{-1} = T_d(s)$ is a proper diagonal nonsingular matrix.

$$\begin{aligned} \therefore\ B^*\ [T_d(s)] &= B^*\ [R(s)\ P_F^{-1}(s)\ G^{-1}] \\ &= B^*\ [R(s)\ [G\ P_F(s)]^{-1}] \\ &= \text{a nonsingular constant matrix} \qquad \text{...(10.7.11)} \end{aligned}$$

But because of (10.7.10)

$$\begin{aligned} &B^*\ [R(s)\ [G\ P_F(s)]^{-1}] \\ &= B^*\ [R(s)\ P^{-1}(s)] \\ &= B^*\ [T(s)] \qquad \text{...(10.7.12)} \end{aligned}$$

Hence from (10.7.11) and (10.7.12), the necessary condition

for diagonalising by l.s.v.f. is given by (10.7.1).

10.8 DYNAMIC DECOUPLING BY L.O.F.

Linear output feedback is the special case of the configuration of Fig. 10.1, where

$$T_u(s) = 0$$

$$T_y(s) = H = \text{a constant matrix} \qquad \ldots(10.8.1)$$

This means,

$$q(s) = 1 \quad \text{and}$$

$$H\,R(s) = F(s) \qquad \ldots(10.8.2)$$

Hence

$$\begin{aligned} P_F(s) &= P(s) - F(s) \\ &= P(s) - H\,R(s) \end{aligned} \qquad \ldots(10.8.3)$$

For diagonalisation, from (10.6.10), (10.6.8) and (10.8.3), assuming invertible T(s)

$$P_d(s)\,\bar{R}(s) = P(s) - H\,R_d(s)\,\bar{R}(s)$$

i.e. $$P(s)\,\bar{R}^{-1}(s) = H\,R_d(s) + P_d(s) \qquad \ldots(10.8.4)$$

Equation (10.8.4) is satisfied if and only if $\bar{R}(s)$ is a unimodular matrix, as then its inverse is a polynomial matrix. From (10.8.4) we can write by post-multiplying both the sides by $R_d^{-1}(s)$,

$$P(s)\,R^{-1}(s) = H + P_d(s)\,R_d^{-1}(s) \qquad \ldots(10.8.5)$$

For satisfying (10.8.5), it is necessary and sufficient that $T^{-1}(s)$ which is equal to the l.h.s. of (10.8.5) must have only scalar numbers in the place of nondiagonal elements. Thus the necessary and sufficient conditions for decoupling

by l.o.f. are :

i) $\bar{R}(s)$ should be a unimodular matrix

ii) $T^{-1}(s)$ must have its nonscalar elements only on the principal diagonal.

10.9 DYNAMIC DECOUPLING : NONINVERTIBLE T(s)

Let us now consider the case when $T(s)$ is of dimension $p \times m$ with $p < m$, satisfying the necessary condition (10.6.2) for diagonalising, namely

$$\rho\left[T(s)\right] = \rho\left[R(s)\ P^{-1}(s)\right]$$
$$= p$$

$$\therefore \quad \rho\left[R(s)\right] = p \qquad \ldots(10.9.1)$$

Let $R_e(s)$ be $m \times m$ matrix obtained by appending $R(s)$ by m-p additional arbitrary rows such that $R_e(s)$ is invertible and

$$d_c\ R_e(s) \leq d_c\ P(s) \qquad \ldots(10.9.2)$$

Then a compensation

$$T_c = P(s)\ P_F^{-1}(s) \qquad \ldots(10.9.3)$$

can be designed for converting $T_e(s)$ to an $m \times m$ desired diagonal matrix $T_{de}(s)$.

$$T_{de}(s) = T_e(s)\ T_c(s) = R_e(s)\ P^{-1}(s)\ T_c(s) \qquad \ldots(10.9.4)$$

Then the $p \times m$ closed loop matrix with given system matrix $T(s)$ and the compensation of (10.9.3) would be

$$T_d(s) = T(s)\ T_c(s) \qquad \ldots(10.9.5)$$

Obviously $T_d(s)$ consists of the first p rows of $T_{de}(s)$. Thus the last m-p columns of $T_d(s)$ would identically be zero and remaining part of $T_d(s)$ of dimension p x p would be diagonal. Thus the p outputs will be affected each by one of the p active inputs and the remaining input will be inactive.

10.10 STATIC DECOUPLING

Static decoupling means the situation in which the steady state value of an output is affected by only one input when all inputs are step inputs applied simultaneously at t = 0. Hence the compensation required for static decoupling is such as to make the closed loop transfer matrix,

$$T_{cl}(s) = R(s)\, P_F^{-1}(s)\, G \qquad \ldots(10.10.1)$$

at s = 0, a diagonal matrix. In case of a square matrix $T(s)$, this can be done by choosing

$$G = \left[R(0)\, P_F^{-1}(0)\right]^{-1}$$

$$= P_F(0)\, R^{-1}(0) \qquad \ldots(10.10.2)$$

This requires that the ranks $R(0)$ and $P_F(0)$ must each be p. As $|P_F(s)|$ is designed a stable polynomial $|P_F(0)|$ is a non-zero scalar. Thus $P_F(0)$ is a nonsingular matrix. Hence

$$\rho\left[R(0)\right] = p \qquad \ldots(10.10.3)$$

is the necessary and sufficient condition for static decoupling. The modification required for determining G in the case of a p x m matrix T(s) with rank p is obvious from Section 10.9 i.e.

$$G = \left[R_e(0)\ P_F^{-1}(0) \right] \qquad \ldots(10.10.4)$$

The necessary and sufficient condition for static decoupling remains the same as (10.10.3). But care must be taken in choosing the last m·p rows of $R_e(s)$ that, in addition to the previous requirements $R_e(0)$ is made invertible.

Exercise 10 :

(10.1) In the illustrative example 10.1 try to specify a $P_F(s)$ with stable poles for the following requirements:

i) The zero order compensation is possible by l.o.f. only.

ii) General zero order compensation of configuration of Fig. 10.1 is possible.

iii) l.s.v.f. is possible. Find the corresponding state feedback matrix F.

(10.2) In the illustrative example 10.1 try to specify the first order q(s) such that the realisation of $T_u(s)$ and $T_y(s)$ is possible by a first order compensator.

(10.3) Given the system transfer matrix

$$T(s)=R(s)\ P^{-1}(s)$$

$$= \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ -1 & s-1 \end{bmatrix}^{-1}$$

The system is to be diagonalised to get the closed loop transfer matrix equal to $R_d(s)\ P_d^{-1}(s)$ where $R_d(s)$ is as defined by equation 10.6.8, using the compensation of Fig. 10.1.

a) What are the maximum possible degrees of the elements of $P_d(s)$?

b) Choose stable polynomials as elements of $P_d(s)$ of the degrees found in (a).

c) Design the corresponding compensation.

(10.4) The given system transfer matrix is

$$T(s) = R(s)\ P^{-1}(s) = \begin{bmatrix} s+1 & s \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s^2 & 0 \\ 0 & s^2 \end{bmatrix}^{-1}$$

Design a feedback compensation to get closed loop transfer matrix $R(s)\ P_F^{-1}(s)$ where

$$P_F'(s) = \begin{bmatrix} (s+1)^2 & s(s+1) \\ (s+2) & (s+2) \end{bmatrix}$$

(10.5) In the illustrative example 10.2, if the complete state feedback is possible, what is the state feedback matrix?

(10.6) Design a static forward compensation G in order to diagonalise the closed loop transfer matrix $R(s)\ P_F^{-1}(s)$ of example 10.4.

(10.7) For the system of problem 9.3 design a compensator of smallest possible order using the frequency domain algorithm of Chapter 10 and its dual. Compare the order with $m(\nu-1)$ and $p(\mu-1)$. Can you utilise algorithm FT_c for converting the triple (A, B, C) into required form of $R(s)\ P^{-1}(s)$?

+ + +

Chapter - 11

Model Matching Compensation

11.1 Introduction

In this chapter a method is described to match the given multivariable system of m inputs and p outputs with a desired model system of q inputs and p outputs. The compensator combines feedback and feedforward compensations to achieve the objective. In general p, q and m can have different values. The algorithm applies to system transfer matrix T(s) of rank p. But it is shown in the following section that in case the rank is not p, the given problem can be modified to have this condition satisfied. Section 11.3 describes the algorithm for designing the compensation and is followed by three illustrative examples.

11.2 Model Matching Problem

It can be stated as follows : The given system has p x m transfer matrix T(s). It is desired to compensate the system to obtain the compensated system transfer matrix, $T_m(s)$.

The effective precompensator transfer matrix $T_c(s)$ required for model matching should be such that,

$$T(s)\, T_c(s) = T_m(s) \qquad \ldots(11.2.1)$$

Let $T(s)$ be written as

$$T(s) = R(s)\; P^{-1}(s) \qquad \ldots(11.2.2)$$

where $R(s)$ and $P(s)$ are relatively right prime and $P(s)$ is column proper. Then 11.2.1 can be written as

$$R(s)\; P^{-1}(s)\; T_c(s) = T_m(s) \qquad \ldots(11.2.3)$$

Consider the case $\rho\,[R(s)] < p$

Let $\rho\,[R(s)] = r$, then we can find a unimodular matrix $U_L(s)$ such that

$$U_L(s)\; R(s) = \begin{bmatrix} R_r(s) \\ --- \\ 0 \end{bmatrix} \qquad \ldots(11.2.4)$$

where $R_r(s)$ is $r \times m$ matrix of rank r.

Let (11.2.3) be premultiplied by $U_L(s)$ giving

$$U_L(s)\; R(s)\; P^{-1}(s)\; T_c(s) = U_L(s)\; T_m(s) \qquad \ldots(11.2.5)$$

$$\begin{bmatrix} R_r(s) \\ --- \\ 0 \end{bmatrix} \quad P^{-1}(s)\quad T_c(s) = \begin{bmatrix} T_{mr}(s) \\ ---- \\ T_{mo}(s) \end{bmatrix} \qquad \ldots(11.2.6)$$

where $T_{mr}(s)$ consists of first r rows of $U_L(s)\; T_m(s)$ and $T_{mo}(s)$ consists of its remaining $(p-r)$ rows. A necessary condition on $T_m(s)$ for model matching to be possible in this case is, obviously,

$$T_{mo}(s) = 0 \qquad \ldots(11.2.7)$$

If this necessary condition is satisfied then the problem can

be converted to achieving the $T_c(s)$, by combination of feedforward and feedback compensations such that

$$T_r(s)\ T_c(s) = T_{mr}(s) \qquad \ldots(11.2.8)$$

where

$$T_r(s) = R_r(s)\ P^{-1}(s) \qquad \ldots(11.2.9)$$

and has its rank equal to its number of rows. Alternatively the given system matrix may have $p \leq m$ and $\rho\,[R(s)] = p$. Thus, in general, it is necessary to solve the model matching problem for $p \leq m$ and $\rho\,[R(s)] = p$.

It is now possible to find out a right inverse $R_{RI}(s)$ of $R(s)$ because

$$\rho\,[R(s)] = p \qquad \ldots(11.2.10)$$

Then to satisfy (11.2.3), a $T_c(s)$ can be found out such that

$$P^{-1}(s)\ T_c(s) = R_{RI}(s)\ T_m(s) \qquad \ldots(11.2.11)$$

i.e. $$T_c(s) = P(s)\ R_{RI}(s)\ T_m(s) \qquad \ldots(11.2.12)$$

One method for finding out $R_{RI}(s)$ is as follows. Amend the p rows of $R(s)$ by (m-p) rows to get $\bar{R}(s)$ which is invertible and preferably $|\bar{R}(s)|$ stable and $P(s)\ \bar{R}^{-1}(s)$ a proper transfer matrix. The latter is assured by selecting the m-p last rows of $\bar{R}(s)$ s.t.

1) $\bar{R}(s)$ is column proper
2) $d_c\ \bar{R}(s) = d_c\ P(s)$ $\qquad \ldots(11.2.13)$

Now, $\bar{R}(s)$ can be related to $R(s)$ by

$$G\,\bar{R}(s) = R(s) \qquad \ldots(11.2.14)$$

where G is a p x m matrix consisting of first p rows of m^{th} order unity matrix. Then substituting (11.2.14) in (11.2.2)

$$T(s) = G\,\bar{R}(s)\,P^{-1}(s) \qquad \ldots(11.2.15)$$

$$\therefore \quad T(s)\,P(s)\,\bar{R}^{-1}(s)\,G^{T} = G\,G^{T} = I_p \qquad \ldots(11.2.16)$$

$$\therefore \quad T_{RI}(s) = P(s)\,\bar{R}^{-1}(s)\,G^{T} \qquad \ldots(11.2.17)$$

Thus for the given $T_m(s)$ the required precompensation can have transfer matrix

$$T_c(s) = P(s)\,\bar{R}^{-1}(s)\,G^{T}\,T_m(s) \qquad \ldots(11.2.18)$$

Let us express $P^{-1}(s)\,T_c(s)$ obtained from (11.2.18) as

$$P^{-1}(s)\,T_c(s) = \frac{R_m(s)}{\phi(s)} \qquad \ldots(11.2.19)$$

where $\phi(s)$ is the least order common denominator polynomial and $R_m(s)$ is the numerator polynomial matrix of order m x q. Such a transfer matrix can, in general, be realised by dynamic feedback combined with a feedforward compensator. The order of feedforward compensation is desired to be as small as possible because of well known advantages of feedback system over feedforward system.

11.3 Combined Dynamic Feedback and Feedforward Compensation

The configuration of the proposed scheme is shown in Fig. 11.3.1. It differs from the configuration of Fig. 10.2.1

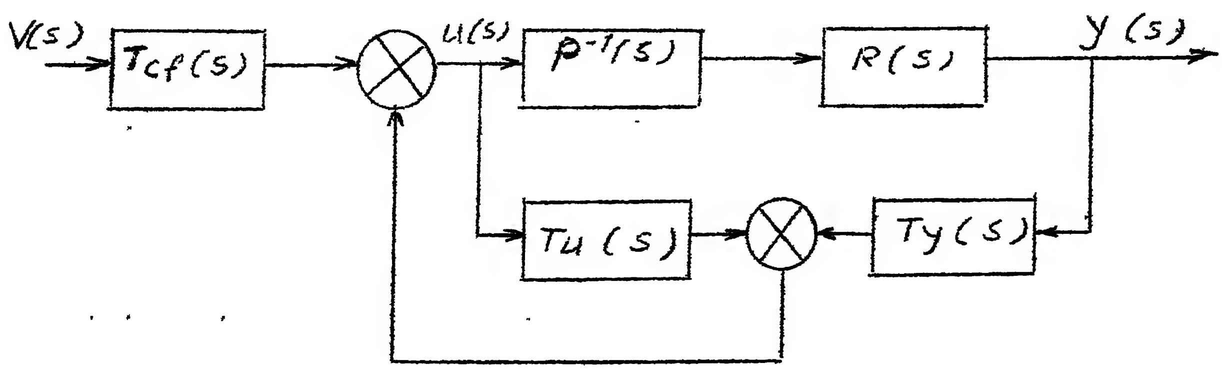

FIG - 11.3.1

only in that the constant forward compensation G is replaced by dynamic forward compensation $T_{cf}(s)$. Hence this scheme can give closed loop transfer matrix,

$$T_{cl}(s) = R(s)\, P_F^{-1}(s)\, T_{cf}(s) \qquad \ldots(11.3.1)$$

where

$$P_F(s) = P(s) - F(s) \qquad \ldots(11.3.2)$$

and F(s) in turn is

$$F(s) = T_u(s)\, P(s) + T_y(s)\, R(s) \qquad \ldots(11.3.3)$$

with $T_u(s)$ and $T_y(s)$ designed such that F(s) is a polynomial matrix as explained in Chapter-10.

From (11.2.1), (11.2.19) and (11.3.1) the desired closed loop transfer matrix $T_m(s)$ can be obtained if,

$$\frac{R_m(s)}{\phi(s)} = P_F^{-1}(s)\ T_{cf}(s) \qquad \ldots(11.3.4)$$

$\therefore$ i.e., from (11.3.2), if

$$\left[P(s) - F(s)\right] R_m(s) = \phi(s)\ T_{cf}(s) \qquad \ldots(11.3.5)$$

(11.3.5) can be written as

$$P(s)\ R_m(s) = F(s)\ R_m(s) + \phi(s)\ T_{cf}(s) \qquad \ldots(11.3.6)$$

Thus the problem of model matching now consists of solving (11.3.6) for $F(s)$ and $T_{cf}(s)$ such that

i) $P_F(s) = P(s) - F(s)$ is invertible

ii) $d_c\ F(s) \leq d_c\ P(s)$

iii) $T_{cf}(s) \neq 0$ $\qquad \ldots(11.3.7)$

Note that the restriction (ii) above is the same as (10.2.8) and is necessary to realise the part $R(s)\ P_F^{-1}(s)$ of $T_m(s)$ by the algorithm of Chapter-10.

Let $P(s)\ R_m(s)$ be written as

$$P(s)\ R_m(s) = \gamma_m\ S_m(s) \qquad \ldots(11.3.8)$$

where

$$S_m(s) = \begin{bmatrix} 1 & 0 & . & 0 \\ s & . & . & . \\ s^2 & . & . & . \\ . & . & . & . \\ . & . & . & . \\ s^{d_{m1}} & 0 & . & . \\ 0 & 1 & . & . \\ . & . & . & 0 \\ . & s^{d_{m2}} & . & 1 \\ . & 0 & . & s \\ . & . & . & . \\ 0 & 0 & . & s^{d_{mq}} \end{bmatrix} \qquad \ldots(11.3.9)$$

where d_{m1}, d_{m2} .. d_{mq} are column degrees of $P(s)\ R_m(s)$. γ_m is a scalar matrix of coefficients. Because of the restriction (ii) of (11.3.7) it is possible to write $F(s)$ as

$$F(s) = F_o + F_1 s + F_2\ s^2 \ldots + F_{r+1}\ s^{r+1} \qquad \ldots(11.3.10)$$

where $r+1$ is equal to the largest of the column degrees d_1, d_2 .. d_m of $P(s)$. $F_o\ F_1 \ldots F_{r+1}$ are $m \times m$ scalar matrices with some of the columns forced to zero in order to satisfy (11.3.7).

Then $F(s)\ R_m(s)$ can be written as;

$$F(s)\ R_m(s) = \left[F_o\ F_1 \ldots F_{r+1}\right] \begin{bmatrix} R_m(s) \\ s\ R_m(s) \\ . \\ . \\ s^{r+1}\ R_m(s) \end{bmatrix}$$

$$\ldots(11.3.11)$$

Eliminate the zero columns of $\begin{bmatrix} F_0 & F_1 & \cdots & F_{r+1} \end{bmatrix}$ and denote the remaining matrix by $\bar{F}$. Similarly eliminate the corresponding rows of

$$\begin{bmatrix} R_m(s) \\ s\, R_m(s) \\ \vdots \\ s^{r+1}\, R_m(s) \end{bmatrix}$$

and denote the remaining matrix by $\bar{R}_m(s)$.

Then

$$F(s)\; R_m(s) = \bar{F}\; \bar{R}_m(s) \qquad \ldots(11.3.12)$$

Let $\bar{R}_m(s)$ be written as

$$\bar{R}_m(s) = \beta_m\; S_m(s) \qquad \ldots(11.3.13)$$

where β_m is a scalar matrix.

Then

$$F(s)\; R_m(s) = \bar{F}\; \beta_m\; S_m(s) \qquad \ldots(11.3.14)$$

Now let us have $T_{cf}(s)$ such that,

$$T_{cf}(s) = \left[\frac{(\, r_{ij}\,)}{g_j(s)} \right] \qquad \ldots(11.3.15)$$

where the element in the matrix sign brackets is the (i, j)th element of $T_{cf}(s)$, with r_{ij}, a scalar and $g_j(s)$ the common denominator monic polynomial of jth column of $T_{cf}(s)$.

Then

$$T_{cf}(s)\; \phi(s) = \left[\frac{r_{ij}}{g_j(s)} \right] \times \phi(s) \qquad \ldots(11.3.16)$$

In order to make $T_{cf}(s)\ \phi(s)$, a polynomial matrix of column degrees equal to or less than the column degrees of $S_m(s)$, select

$$g_j(s) = 1, \text{ when } d_\phi \leq d_{mj} \qquad \ldots(11.3.17)$$

where d_ϕ is the degree of $\phi(s)$ and d_{mj} is the degree of jth column of $S_m(s)$. Similarly select

$$g_j(s) = \text{a polynomial of degree } (d_\phi - d_{mj}) \text{ which is a factor of } \phi(s),$$

when

$$d_\phi > d_{mj} \qquad \ldots(11.3.18)$$

If necessary, select for $g_j(s)$ a factor of $\phi(s)$ of degree, nearest larger to $(d_\phi - d_{mj})$. Then in any case (11.3.17) or (11.3.18),

$$T_{cf}(s)\ \phi(s) = [R_{cf}] \begin{bmatrix} \phi_1(s) & 0 & \cdot & \cdot \\ 0 & \phi_2(s) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \phi_q(s) \end{bmatrix}$$

where, $[R_{cf}] = [r_{ij}]$ $\qquad \ldots(11.3.19)$

and the jth diagonal element $\phi_j(s)$ of (11.3.19) is a polynomial of degree less than or equal to d_{mj} given by

$$\phi_j(s) = \frac{\phi(s)}{g_j(s)} \qquad \ldots(11.3.20)$$

Equation (11.3.19) can be expressed as

$$T_{cf}(s)\,\phi(s) = R_{cf}\,\alpha_m \cdot S_m(s) \qquad \ldots(11.3.21)$$

Now using (11.3.8), (11.3.14) and (11.3.21), the equation (11.3.6) can be written as

$$\gamma_m\,S_m(s) = \bar{F}\,\beta_m\,S_m(s) + R_{cf}\,\alpha_m\,S_m(s) \qquad \ldots(11.3.22)$$

which can further be written as

$$\left[\alpha_m^T \;;\; \beta_m^T\right] \begin{bmatrix} R_{cf}^T \\ \bar{F}^T \end{bmatrix} = \gamma_m^T \qquad \ldots(11.3.23)$$

This equation can be solved for R_{cf} and $\bar{F}$ if and only if

$$\rho\left[\alpha_m^T \;;\; \beta_m^T\right] = \rho\left[\alpha_m^T \;;\; \beta_m^T \;;\; \gamma_m^T\right] \qquad \ldots(11.3.24)$$

Assuming (11.3.24) is satisfied, we can solve for R_{cf} and $\bar{F}$. The solution is acceptable only if it gives corresponding $P_F(s)$ satisfying (11.3.7). If (11.3.24) is not satisfied or if $P_F(s)$ is not in the desired form, second trial has to be done. The trial and error element is involved in determining $R_{RI}(s)$ and in selecting $g_j(s)$. The solution of (11.3.23) can be surely made to satisfy the necessary requirements (11.3.7) if $F(s)$ is restricted to have

$$d_c\,F(s) < d_c\,P(s) \qquad \ldots(11.3.25)$$

Correspondingly the last term of (11.3.10) will be $F_r\,s^r$. With this restriction, an additional advantage obtained is that the dynamic feedback compensation ~~can~~ readily be converted to static state feedback and a zero order feedback compensation can be used if the complete state is available for feedback. The conversion can be obtained by using the relation

$$F(s) = F\,S(s) \qquad \ldots(11.3.26)$$

where S(s) is a matrix defined by (7.4.6) with its column degrees equal to $d_j - 1$ where d_j are the column degrees of P(s) for j = 1 to m. F gives the static state feedback matrix.

But if the equation (11.3.23) can be solved obtaining $P_F(s)$ in the required form, without restriction (11.3.25) i.e. by allowing

$$d_c\, F(s) \leq d_c\, P(s) \qquad \ldots(11.3.27)$$

it helps in reducing the order of forward compensator $T_{cf}(s)$, as illustrated by one of the numerical examples illustrated in the following.

Example 11.1 : Let us consider a practical system. The symplified mathematical model of a single spool turbojet engine can be given by the following 2nd order state space description[(S-5)] with two inputs and two outputs.

$$\dot{X} = A\,x + Bu$$
$$y = C\,X + Eu$$

where

x_1 = roter speed

x_2 = tail pipe pressure

y_1 = thrust

y_2 = turbine temperature

u_1 = fuel flow rate

u_2 = exhaust nozzle area

Let us assume, for convenience of computation, that A, B, C, E for an equivalent system are given by

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The corresponding transfer matrix is

$$T(s) = \begin{bmatrix} \frac{s+1}{s+2} & \frac{s}{s-2} \\ \frac{1}{s+2} & \frac{3}{s-2} \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & s \\ 1 & 3 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s-2 \end{bmatrix}^{-1}$$

$$= R(s)\, P^{-1}(s) \text{ say}$$

with R(s) and P(s) relatively right prime and P(s) column proper.

Let

$$T_m(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

be the desired compensated system transfer matrix. Then the compensation can be designed as follows :

$$P^{-1}(s)\, T_c(s) = R^{-1}(s)\, T_m(s)$$

$$= \frac{1}{2s^2 + 5s+3} \begin{bmatrix} 3(s+1) & -s \\ -(s+1) & s+1 \end{bmatrix}$$

$$= \frac{1}{\phi(s)} R_m(s)$$

$$\therefore \quad P(s)\, R_m(s) = \begin{bmatrix} s+2 & 0 \\ 0 & s-2 \end{bmatrix} \begin{bmatrix} 3(s+1) & -s \\ -(s+1) & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3s^2 + 9s+6 & -s^2 - 2s \\ -s^2 + s+2 & s^2 - s - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 & 3 & 0 & -2 & -1 \\ 2 & 1 & -1 & -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix}$$

$$= \gamma_m\, S_m(s)$$

Let $T_{cf}(s) = R_{cf}$

$$\therefore \quad \phi(s)\, I_2 = \begin{bmatrix} 2s^2 + 5s+3 & 0 \\ 0 & 2s^2 + 5s+3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 5 & 2 \end{bmatrix} S_m(s)$$

$$= \alpha_m\, S_m(s)$$

Let $\overline{F} = F_o$

$$\overline{R}_m(s) = R_m(s) = \beta_m \, S_m(s)$$

$$\therefore \quad \beta_m = \begin{bmatrix} 3 & 3 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We observe that

$$\rho\begin{bmatrix} \alpha_m^T & ; & \beta_m^T \end{bmatrix} = 4$$

and $$\rho\begin{bmatrix} \alpha_m^T & ; & \beta_m^T & ; & \gamma_m^T \end{bmatrix} = 4$$

Hence the unique solution to equation is,

$$\begin{bmatrix} R_{cf} \\ \overline{F}^T \end{bmatrix}^T = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \\ 1 & 0 \\ 1.5 & -3.5 \end{bmatrix}$$

i.e. $$T_{cf}(s) = R_{cf} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1.5 \\ 0 & -3.5 \end{bmatrix}$$

which, it is possible to implement by static state feedback if the same is available.

$$P_F(s) = P(s) - F$$

$$= \begin{bmatrix} s+1 & -1.5 \\ 0 & s+1.5 \end{bmatrix} \text{ is stable and invertible}$$

$\therefore$ The closed loop transfer matrix is

$$T_{cl}(s) = R(s)\, P_F^{-1}(s)\, T_{cf}(s)$$

$$= \frac{\begin{bmatrix} s+1 & s \\ 1 & 3 \end{bmatrix} \begin{bmatrix} s+1.5 & 1.5 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}}{(s+1)\,(s+1.5)}$$

$$= \begin{bmatrix} s+1 & s \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1.5(s+1) & -0.5s \\ -0.5(s+1) & 0.5(s+1) \end{bmatrix} \frac{1}{(s+1)\,(s+1.5)}$$

$$= \begin{bmatrix} (s+1)(s+1.5) & 0 \\ 0 & (s+1.5) \end{bmatrix} \frac{1}{(s+1)\,(s+1.5)}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

as desired.

Example 11.2 :

The given system is described by the following transfer matrix

$$T(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{s+3}{s+2} \\ \frac{1}{s+1} & 0 \end{bmatrix}$$

$$= R(s)\, P^{-1}(s)$$

$$= \begin{bmatrix} s+2 & s+3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1}$$

with R(s) and P(s) relatively right prime and P(s) column proper.

Objective : To get the closed loop transfer matrix $T_{cl}(s)$ equal to the model transfer matrix.

(1) $T_m(s) = \begin{bmatrix} \frac{s+1}{s+4} \\ \frac{-2}{(s+2)(s+4)} \end{bmatrix}$ (2) $T_m(s) = \begin{bmatrix} \frac{1}{s+4} \\ \frac{-2}{(s+2)(s+4)} \end{bmatrix}$

This model matching problem is solved by Silverman[(s-1)] and Tzafestas[(T-1)] by using other methods. The application of the present algorithm to this problem is illustrated below.

Solution :

(i) As T(s) is invertible, desired $T_c(s)$ is

$$T_c(s) = T^{-1}(s)\ T_m(s)$$

$$= P(s)\ R^{-1}(s)\ T_m(s)$$

i.e. $$P^{-1}(s)\ T_c(s) = R^{-1}(s)\ T_m(s)$$

$$= \begin{bmatrix} s+2 & s+3 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{s+1}{s+4} \\ \frac{-2}{(s+2)(s+4)} \end{bmatrix}$$

$$= \frac{1}{(s+2)(s+4)} \begin{bmatrix} -2 \\ s+2 \end{bmatrix}$$

Equation (11.3.5) can be written as,

$$\left[P(s) - F(s)\right] \begin{bmatrix} -2 \\ s+2 \end{bmatrix} = (s+2)\ (s+4)\ T_{cf}(s)$$

i.e. $$\begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} -2 \\ s+2 \end{bmatrix} = F(s) \begin{bmatrix} -2 \\ s+2 \end{bmatrix} + T_{cf}(s)(s+2)(s+4).$$

As 11.3.17 is satisfied for $j = 1 = q$, $T_{cf}(s) = R_{cf}$

$\bar{F} = F_o$ as each column degree of $P(s)$ is 1.

$$S_m(s) = \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}$$

Equation 11.3.23 can be written as

$$\begin{bmatrix} 8 & . & -2 & 2 \\ 6 & . & 0 & 1 \\ 1 & . & 0 & 0 \end{bmatrix} \begin{bmatrix} R_{cf}^{\ T} \\ F_o^{\ T} \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -2 & 4 \\ 0 & 1 \end{bmatrix}$$

$$\rho\left[\alpha_m^{\ T} \ ; \ \beta_m^{\ T}\right] = 3 = \text{full rank} = \rho\left[\alpha_m^{\ T} \ ; \ \beta_m^{\ T} \ ; \ \gamma_m^{\ T}\right]$$

$\therefore$ The above equation can be solved to give unique solution,

$$\begin{bmatrix} R_{cf}^{\ T} \\ F_o^{\ T} \end{bmatrix} = \begin{bmatrix} 8 & -2 & 2 \\ 6 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 4 \\ -2 & 4 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -2 & -2 \end{bmatrix}$$

(2) For the $T_m(s)$ specified in (2),

$$R^{-1}(s)\,T_m(s) = P^{-1}(s)\,T_c(s)$$

$$= \frac{1}{(s+2)(s+3)(s+4)} \begin{bmatrix} -2(s+3) \\ 3(s+2) \end{bmatrix}$$

As 11.3.18 is true in this case for $j = 1 = q$, let

$$T_{cf}(s) = R_{cf}\,\frac{1}{s+4}$$

where, $g_1(s) = s+4$

Then, 11.3.23 can be written as

$$\begin{bmatrix} 6 & -6 & 6 \\ 5 & -2 & 3 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_{cf}^{\ T} \\ F_o^{\ T} \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ -8 & 12 \\ -2 & 3 \end{bmatrix}$$

$$\therefore\quad R_{cf} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \text{and} \quad F_o = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore\quad T_{cf}(s) = \begin{bmatrix} \dfrac{-2}{s+4} \\[2ex] \dfrac{3}{s+4} \end{bmatrix}$$

Example 11.3 : Let the system be described by the state space quadruple :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 0 \end{bmatrix}$$

Then,

$$T(s) = \begin{bmatrix} \frac{s+1}{s^2} & \frac{s+2}{s^2+1} \\ \frac{2}{s} & \frac{2s+3}{s^2+1} \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & 1 \\ 2s & 3 \end{bmatrix} \begin{bmatrix} s^2 & -s^2 \\ 0 & s^2+1 \end{bmatrix}^{-1}$$

$$= R(s)\ P^{-1}(s)$$

with $R(s)$ and $P(s)$ relatively right prime and $P(s)$ column proper.

Let the desired closed loop transfer matrix be

$$T_m(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}$$

Then the compensation can be designed as follows :

$$R^{-1}(s)\ T_m(s) = \frac{\begin{bmatrix} 3 & -(s+1) \\ -2s & s^2+\ 2s+1 \end{bmatrix}}{s^2+\ 4s\ +\ 3}\ .$$

$$= \frac{R_m(s)}{\phi(s)}$$

$$S_m{}^T(s) = \begin{bmatrix} 1 & s & s^2 & s^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & s & s^2 & s^3 & s^4 \end{bmatrix}$$

Then

$$\gamma_m = \begin{bmatrix} 0 & 0 & 3 & 2 & 0 & 0 & -2 & -3 & -1 \\ 0 & -2 & 0 & -2 & 1 & 2 & 2 & 2 & 1 \end{bmatrix}$$

$$\alpha_m = \begin{bmatrix} 3 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 4 & 1 & 0 & 0 \end{bmatrix}$$

$$\beta_m = \begin{bmatrix} 3 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

The above β_m corresponds to

$$\bar{F} = \begin{bmatrix} F_0 & F_1 & F_2 \end{bmatrix}$$

It can be found that

$$\rho\begin{bmatrix} \alpha_m^T & ; & \beta_m^T \end{bmatrix} = \rho\begin{bmatrix} \alpha_m^T & ; & \beta_m^T & ; & \gamma_m^T \end{bmatrix} < 8$$

Hence nonunique solution is possible.

One solution is

$$\begin{bmatrix} R_{cf}^T \\ \bar{F}^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ -1 & -2 \\ -2 & -2 \\ -1 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$$

Hence

$$T_{cf}(s) = R_{cf} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $$F(s) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ -2 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} s^2$$

$$= \begin{bmatrix} -1-2s & -(1+s+s^2) \\ -2s & -2+s^2 \end{bmatrix}$$

This $F(s)$ can be realised by transfer matrices, $T_u(s)$ and $T_y(s)$, using algorithm of Chapter-10.

Exercise 11 :

(11.1) In a model matching problem,

i) $T_m(s)$ is invertible

ii) $T_m^{-1}(s)$ is a polynomial matrix

iii) $R(s)$ is invertible

and iv) $d_c\left[T_m^{-1}(s)\ R(s)\right] > d_c\left[P(s)\right]$, for one or more columns.

Prove that a constant forward compensation in combination with feedback compensation using the algorithm described in Chapter 11 is not possible in this case.

(11.2) For a system described by

$$T(s) = \begin{bmatrix} s+1 & 1 \\ 2s & 3 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 0 & s-2 \end{bmatrix}^{-1}$$

The desired model transfer matrix is

$$T_m(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

Design a combined feedforward and feedback compensation. Are the conditions in problem 11.1 satisfied in this case?

(11.3) For a system described by

$$T(s) = \begin{bmatrix} s+1 & 1 \\ 2s & 3 \end{bmatrix} \begin{bmatrix} s^2 & -s \\ 0 & s-2 \end{bmatrix}^{-1}$$

design a feedback feedforward compensation to achieve the compensated system transfer matrix,

$$T_m(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

(11.4) The transfer matrix of a system is given by

$$T(s) = \begin{bmatrix} \frac{-2}{s-2} \\ \frac{3}{s-3} \end{bmatrix}$$

Find out whether it is possible to design a compensation to achieve the compensated system transfer matrix,

$$T_m(s) = \begin{bmatrix} \frac{2(s-3)}{3(s+1)} \\ -\frac{(s-2)}{(s+1)} \end{bmatrix}$$

Design the compensation if possible.

(11.5) The transfer matrix of a system is given by

$$T(s) = R(s)\ P^{-1}(s)$$

$$= \begin{bmatrix} s+1 & -s \end{bmatrix} \begin{bmatrix} s^2+1 & 0 \\ 0 & s-2 \end{bmatrix}^{-1}$$

Design a compensation to achieve the compensated system transfer matrix,

$$T_m(s) = \frac{1}{s+1}$$

+ + +

CHAPTER - 12

REVIEW AND COMMENTS

12.1 Introduction

Looking back at what has been covered in last 11 chapters, it can be noticed that this gives a complete in itself treatment of basic analysis and synthesis problems in multivariable control systems. The specific approach accepted for this book limits the number of techniques presented in detail. The specific choice of techniques is based on the following points :

i) Generalised system description, having no restrictions on number of inputs m, number of outputs p and system order n.

ii) Less amount of trial and error element.

iii) More insight with less complexity of computation.

iv) Systematic logical sequence of development.

v) Practical orientation.

A major line of development omitted from this book consists of the methods based on direct extension of classical control techniques to mvcs. Following features more or less common to all such methods restrict their use to particular type of problem with some limitations on design freedom :

i) The no. of inputs is assumed equal to the no. of outputs.

ii) The basic principle is to remove, by diagonalisation or by diagonal dominance, the interaction between different

subsystems and then treat each of the subsystems as a scalar system to which classical techniques like Nyquist plot, Root loci, Routh Hurwitz criterion etc. can be applied. This procedure basically tends to increase the amount of labour.

iii) As in classical control theory the effort is directed towards seeking the range of stability as a function of variation of some parameters of the system and then choosing particular values of these parameters for satisfying the design specification. It seems preferable to choose, directly, the values of all possible parameters to satisfy the design specification simultaneously taking care of the system stability.

iv) Non-interaction, being the basic requirement restricts the structure of feedback and compensation transfer matrices reducing the design freedom.

12.2 Extension of Classical Theory to MVCS

In spite of the restrictions listed above these methods extending the classical theory to mvcs are useful for particular problems and have been applied for practical problems. Hence a brief description of the following methods of this kind is included in this chapter.

i) Commutative Controller

ii) Diadic Transfer Matrices

iii) Inverse Nyquist Array

iv) Sequential Design

v) MV Root Loci.

The system configuration assumed in all these methods is given in Fig. 12.2.1. The feedback transfer matrix H(s) is assumed equal to unity matrix, in most of the cases.

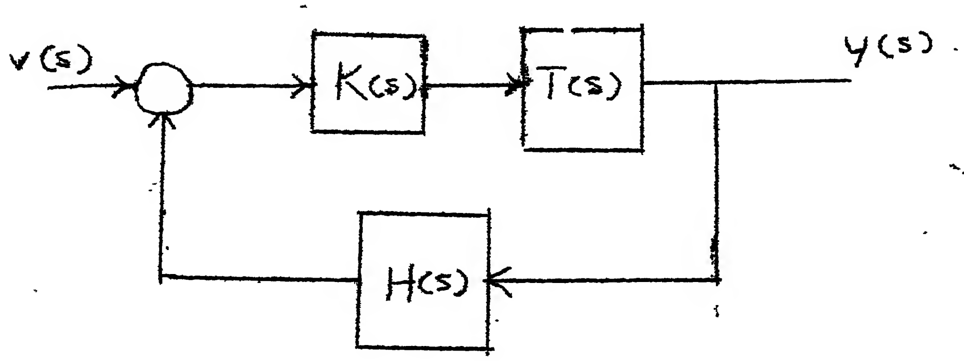

FIG. 12.2.1

K(s) is the controller transfer matrix to be designed. T(s) is a p x p transfer matrix describing the uncompensated system. The closed loop transfer matrix is given by

$$T_{cl}(s) = [I + T(s)\ K(s)\ H(s)]^{-1}\ T(s)\ K(s) \qquad ...(12.2.1)$$

The return difference matrix is

$$\Delta(s) = I + G(s)\ H(s) \qquad ...(12.2.2)$$

where

$$G(s) = T(s)\ K(s) \qquad ...(12.2.3)$$

The characteristic equation is obtained by equating the determinant of $\Delta(s)$ to zero, i.e.

$$|\ I + T(s)\ K(s)\ H(s)\ | = 0 \qquad ...(12.2.4)$$

Thus, for stability, the zeros of $|\Delta(s)|$ are required to be in open left half of s plane (o.l.h.p.).

Suppose $T(s)$ is a diagonal matrix given by

$$T(s) = \text{diag}\left[t_1(s)\ t_2(s)\ \dots\ t_p(s)\right] \qquad \dots(12.2.5)$$

where any $t_i(s)$ is a transfer function element. Then the multivariable design becomes equivalent to p SISO designs consisting of designing the p compensating transfer functions $k_1(s)$, $k_2(s)$ .. $k_p(s)$ of a diagonal compensator $K(s)$, assuming $H(s) = I$. If this simple concept is to be used to extend the classical theory to mvcs it is first necessary to have the system matrix in the diagonal or diagonally dominant form. Naturally conversion of given $T(s)$ to such a form becomes a major step of the methods listed above. Highlights of these methods are given in the following sections.

12.3 Commutative Controller

This method is deviced by Macfarlane[M-6] and can be used for a particular type of p x p transfer matrix $T(s)$, which can be expressed as

$$T(s) = E(s)\ T_d(s)\ E^{-1}(s) \qquad \dots(12.3.1)$$

$E(s)$ is the matrix of eigenvectors and $T_d(s)$ is the diagonal matrix of transfer function eigen elements of $T(s)$. It is not in general possible to have decomposition of $T(s)$ of the type shown in 12.3.1 with rational function elements in $E(s)$ and $T_d(s)$. Hence this is a limitation on its applicability. If however $T(s)$ is of this type, the system can be designed for $T_d(s)$. The feedback matrix $H_d(s)$ and controller matrix $K_d(s)$

can be designed diagonal. This is equivalent to p independent SISO system designs. The controller and feedback transfer matrices for the original system can then be worked out as

$$K(s) = E(s)\ K_d(s)\ E^{-1}(s)$$

and $$H(s) = E(s)\ H_d(s)\ E^{-1}(s) \qquad \ldots(12.3.2)$$

giving the return difference matrix

$$I + T(s)\ K(s)\ H(s) = E(s)\left[I+T_d(s)K_d(s)H_d(s)\right] E^{-1}(s) \qquad \ldots(12.3.3)$$

Again it cannot be guaranteed that $I + T(s) \cdot K(s)\ H(s)$ will have desired modes of response because of satisfactory design of $I + T_d(s)\ K_d(s)\ H_d(s)$, because this term is dynamically affected by premultiplication by $E(s)$ and post-multiplication of $E^{-1}(s)$.

Practical diagonalisation of $T(s)$ as per 12.3.1 i.e. realisation of precompensation $E^{-1}(s)$ and postcompensation $E(s)$ is obviously difficult and undesirable and hence is not considered as a possible alternative for overcoming the above mentioned design difficulties and limitations.

Any rational transfer matrix $T(s)$ can be diagonalised as

$$T_d(s) = M(s)\ T(s)\ N(s) \qquad \ldots(12.3.4)$$

as pointed out by Allwright[(A-1)].

Here $M(s)$ and $N(s)$ are rational and invertible. But other difficulties mentioned above are still applicable to this type

of diagonalisation also. A more desirable situation would be to have M and N as constant matrices in 12.3.4. But then this is not possible in general.

12.4 Diadic Transfer Matrix (DTM)

This method due to Owens(O-2) is applicable to a dyadic transfer matrix which by definition can be expressed as

$$T(s) = M.T_d(s).N \qquad \ldots(12.4.1)$$

where M and N are scalar invertible matrices. If there is a constant c for which T(c) is finite and nonsingular then the matrix $T(s)\ T^{-1}(c)$ has rational eigenvalues given by $\dfrac{t_i(s)}{t_i(c)}$. This can be easily verified by substituting from 12.4.1 in T(s) T(c). Thus by a scalar precompensator $T^{-1}(c)$ the dyadic system transfer matrix T(s) can be modified for making it convenient to apply commutative controller design. Diagonal $K_d(s)$ and $H_d(s)$ can thus be designed for the diagonalised $T_d(s)$. The Nyquist plots of the corresponding p scalar subsystems are called characteristic Loci.

In this case the post and premultiplication matrices converting the properly designed $K_d(s)$ and $H_d(s)$ to K(s) and H(s) would be scalar matrices. Thus they do not introduce additional dynamics. However they do create interaction and the satisfactory design for the decoupled system may not prove satisfactory in the presence of such interaction.

12.5 Inverse Nyquist Array (INA)

This method is due to Rosenbrock(R-3). Assuming H(s) = I

and using equations 12.2.1 and 12.2.3 the closed loop transfer matrix can be written as

$$T_{cl}(s) = [I + G(s)]^{-1} \; G(s) \qquad \ldots(12.5.1)$$

The poles of $T_{cl}(s)$ are the zeros of the transfer function $|I + G(s)|$. The closed loop system is stable if these zeros are in the l.h.s. of s-plane. From 12.5.1

$$|I + G(s)| = \frac{|T_{cl}^{-1}(s)|}{|G^{-1}(s)|} \qquad \ldots(12.5.2)$$

and

$$[T_{cl}^{-1}(s)] = G^{-1}(s) \; [I + G(s)]$$

$$= G^{-1}(s) + I$$

$$\therefore \quad |T_{cl}^{-1}(s)| = |G^{-1}(s) + I| \qquad \ldots(12.5.3)$$

$\therefore$ From 12.5.2 and 12.5.3

$$|I + G(s)| = \frac{|G^{-1}(s) + I|}{|G^{-1}(s)|} \qquad \ldots(12.5.4)$$

Assuming G(s) has no poles in the r.h.s., the system will be stable if the Nyquist plot of $|I + G(s)|$ has no encirclements of origin.

Let the encirclements of origin by the Nyquist plot of a transfer function f(s) be denoted by $E_n[f(s)]$. Then from 12.5.4, the stability condition can be written as

$$E_n \; [|G^{-1}(s) + I|] = E_n \; [|G^{-1}(s)|] \qquad \ldots(12.5.5)$$

Hence the controller transfer matrix K(s) has to be designed to satisfy 12.5.5. Had the given T(s) been diagonal, then satisfying 12.5.5 would have been equivalent to classical design of p SISO systems using Nyquist criterion. But there are difficulties in diagonalisation as stated in the previous two methods. In this method of Rosenbrock attempt is made to make the matrices $[G^{-1}(s) + I]$ and $[G^{-1}(s)]$, diagonally dominant. A matrix A(s) is said to be diagonally dominant over a range of s if and only if

$$|a_{ii}(s)| > \sum_{\substack{j=i \\ j\neq i}}^{p} |a_{ij}(s)| \qquad \ldots(21.5.6)$$

for i = 1, 2 .. p and for the specified range of s. $a_{ij}(s)$ in 12.5.6 is the (i, j)th transfer function element of A(s).

Following theorem for a diagonally dominant transfer matrix A(s), based on Gershgorin's theorem[R-4], makes it possible to apply Nyquist criterion to mvcs. Layton[L-2] may be referred for the proof of the theorem.

Theorem :

If a square rational transfer matrix A(s)

i) is diagonally dominant over a curve c in s-plane and

ii) none of the elements $a_{ii}(s)$ has any pole on c

then

$$E_n [|A(s)|] = \sum_{i=1}^{p} E_n [a_{ii}(s)] \qquad \ldots(12.5.7)$$

Thus if we can make $[G^{-1}(s)]$ and $[G^{-1}(s) + I]$ diagonally dominant over the Nyquist path in s-plane then because of 12.5.7, it is possible to satisfy 12.5.5 by satisfying for i = 1,2 ... p,

$$E_n [g_{ii}(s)] = E_n [g_{ii}(s) + i] \qquad \ldots(12.5.8)$$

Thus the given problem becomes equivalent to p scalar system designs. The design now consists of selecting the controller transfer matrix K(s) s.t.

i) There are no poles of K(s) in r.h.s. in order to have no poles of G(s) in r.h.s. assuming T(s) to satisfy this condition.

ii) K(s) makes $G^{-1}(s)$ d.d. over a path c in the r.h.s. of s-plane sufficiently large to enclose all the zeros of $|G(s) + I|$, if any.

iii) K(s) makes $[G^{-1}(s) + I]$ also diagonally dominant over c.

iv) K(s) is such as to satisfy (12.5.8) ...(12.5.9)

In order to choose such a K(s) systematically, it is expressed as

$$K(s) = K_a \, K_b(s) \, K_c(s) \qquad \ldots(12.5.10)$$

K_a is a scalar matrix representing column operations on T(s), to achieve possible amount of diagonal dominance. $K_b(s)$ is chosen to achieve (ii) and (iii) of 12.5.9 and or diagonal matrix $K_c(s)$ is designed to achieve (iv) of 12.5.9. But this

task of selecting $K(s)$ is quite difficult because of the following reasons :

a) There is no proof of existence of a $K_b(s)$ of the required type mentioned above.

b) In case it is possible, generally a lot of trial and error is required.

c) The diagonal dominance of the matrices under consideration has to be checked at a sufficiently large number of points on c.

d) The diagonal dominance may require restriction on the values of s.

The resulting restriction on c may make it difficult to assure whether the region enclosed by c is sufficiently large to enclose the possible zeros of $|I + G(s)|$ in r.h.s., which is a necessary requirement for applying Nyquist criterion.

12.6 Sequential Design

This method is devised by Mayne(M-7). Referring to the block diagram 12.2.1 in this method $H(s)$ is a unity matrix. The feedback matrix has the form,

$$K(s) = K_a \, K_b(s) \, K_c(s) \qquad \ldots(12.6.1)$$

Then

$$\begin{aligned} G(s) &= T(s) \, K_a . K_b(s) . K_c(s) \\ &= \overline{G}(s) \, K_c(s) \end{aligned} \qquad \ldots(12.6.2)$$

where

$$\overline{G}(s) = T(s) . K_a . K_b(s) \qquad \ldots(12.6.3)$$

K_a is a scalar matrix representing column operations on the system matrix T(s) to achieve as much diagonal dominance as possible. $K_b(s)$ and $K_c(s)$ are designed sequentially. At any q^{th} stage in the sequential design $K_c(s) = K_{cq}(s)$ is given by

$$K_{cq}(s) = \text{diag.}\left[k_{1c}(s)\ k_{2c}(s)\ \ldots\ k_q(s)\ 0\ ..\ 0\right] \qquad \ldots(12.6.4)$$

where $k_{ic}(s)$ is a transfer function element. At q^{th} stage, $K_b(s) = K_{bq}(s)$ is given by

$$K_{bq}(s) = K_{1b}(s)\ K_{2b}(s)\ \ldots\ K_{qb}(s) \qquad \ldots(12.6.5)$$

where $K_{ib}(s)$ is a p x p transfer matrix of the form

$$K_{ib}(s) = \begin{bmatrix} I_{i-1} & 0 \\ 0 & K_{p-i+1} \end{bmatrix} \qquad \ldots(12.6.6)$$

Then

$$\bar{G}_q(s) \triangleq T(s)\ K_a.K_{bq}(s) \qquad \ldots(12.6.7)$$

and

$$\Delta_q(s) \triangleq I + \bar{G}_q(s)\ K_{cq}(s) \qquad \ldots(21.6.8)$$

$$\bar{G}_q(s) = \left[T(c).K_a.K_{1b}(s).K_{2b}(s)\ \ldots\ K_{(q-1)b}(s)\right] K_{qb}(s)$$

$$= \bar{G}_{q-1}(s).K_{qb}(s) \qquad \ldots(12.6.9)$$

$$\therefore \quad \bar{G}_q(s) = \bar{G}_{q-1}(s) \begin{bmatrix} I_{q-1} & 0 \\ 0 & K_{p-q+1} \end{bmatrix} \qquad \ldots(12.6.10)$$

$$\therefore \quad \bar{G}_q(s) = \Big[[\bar{G}_{q-1}(s)]_{*1} \cdots [\bar{G}_{q-1}(s)]_{*(q-1)} [\bar{G}_q(s)]_{*q} \cdots [\bar{G}_q(s)]_{*p} \Big] \qquad \ldots(12.6.12)$$

where the suffix *j denotes the j^{th} column of the matrix.

$$\therefore \quad \bar{G}_q(s)\, K_{cq}(s) = \bar{G}_q(s) \begin{bmatrix} k_{1c}(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{2c}(s) & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & k_{cq}(s) & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}$$

$$= \Big[k_{1c}(s).[\bar{G}_{q-1}(s)]_{*1} \cdots k_{(q-1)}(s)\, [\bar{G}_{q-1}(s)]_{*(q-1)} \; k_{qc}(s)\, [\bar{G}_q(s)]_{*q}\; 0..0 \Big]$$

$$= \bar{G}_{q-1}(s).K_{c(q-1)}(s) + k_{qc}(s).\bar{G}_q(s).I_{*q}.I_{q*} \qquad \ldots(12.6.13)$$

Now, substituting 12.6.13 in 12.6.8,

$$\Delta_q(s) = I + \bar{G}_q(s).K_{cq}(s)$$

$$= I + \bar{G}_{q-1}(s).K_{c(q-1)}(s) + k_{qc}(s).\bar{G}_q(s).I_{*q}.I_{q*}$$

$$= \Delta_{q-1}(s)\, [I + k_{qc}(s).\Delta_{q-1}^{-1}\, \bar{G}_q(s)\, I_{*q}\, I_{q*}] \qquad \ldots(12.6.14)$$

The right side factor of the right hand side of 12.6.14 is a unity matrix except its q^{th} column which is equal to $k_{qc}(s)$ times the q^{th} column of $\Delta_{q-1}^{-1}\, \bar{G}_q(s)$. Hence

$$|\Delta_q(s)| = |\Delta_{q-1}(s)|.(1 + k_{qc}(s)\ g_{qq}(s)) \qquad ...(12.6.15)$$

where $g_{qq}(s)$ is the $(q,q)^{th}$ element of $\overset{-1}{\underset{q-1}{}}\ \overline{G}_q(s)$. Let

$$\delta_q = 1 + k_{qc}(s).g_{qq}(s) \qquad ...(12.6.16)$$

$$\begin{aligned} \therefore \quad |\Delta_q(s)| &= |\Delta_{q-1}(s)|.\delta_q \\ &= |\Delta_{q-2}(s)|.\delta_{q-1}.\delta_q \\ &= \ .\ .\ .\ .\ . \\ &= \delta_1\ \delta_2\ \ldots\ \delta_q \end{aligned} \qquad ...(12.6.17)$$

as $|\Delta_o| = 1$, which can be verified easily.

Thus for having stability it is necessary to design each δ_q for $q = 1, 2 \ldots p$ to have its zeros in the open left half plane (o.l.h.p.). From 12.6.16 it can be achieved by proper design of $k_{qc}(s)$. It is desirable to have gain of $k_{qc}(s)$ quite high in order to achieve specifications like accuracy, insensitivity to parameter variation and disturbance, reducing interaction etc. Hence $k_{qc}(s)$ is designed to make δ_q stable for as high gain as possible. This however makes

$$\delta_q \cong k_{qc}(s).g_{qq}(s) \qquad ...(12.2.18)$$

Then it is necessary to have the zeros of $k_{qc}(s)$ and $g_{qq}(s)$ also in the o.l.h.p. $k_{qc}(s)$ can be designed to take this factor into account. For achieving zeros of $g_{qq}(s)$ in o.l.h.p., k_{qb} can be used. Note that only the q^{th} column of K_{qb} affects

$g_{qq}(s)$ and it does not affect the design of previous stages, because of its particular structure given by 12.6.6.

12.7 Root Loci in MVCS

The numerator of the transfer function $|I + T(s)\ K(s)H(s)|$ can be taken as the characteristic polynomial of the mvcs. Root loci can be drawn for this polynomial with some characteristic gains as variable parameters. Generally it is difficult to consider the root loci for all the p^2 gains of the p^2 transfer function elements. Hence for any $T(s)\ K(s)\ H(s)$ where the $K(s)$ and $H(s)$ are designed by some other method, the root loci are drawn for $[I + k\ T(s)\ k(s)\ H(s)]$ where k is the scalar gain parameter for which the root loci are drawn. Or, the root loci for mvcs are obtained as combinations of the root loci for the exactly or approximately decoupled subsystems.

Detailed discussion on this subject is available in the book of Owens[O-4] and other publications[O-3,K-6]. The main difference in this method and those discussed in the previous sections of this chapter is that in this case the classical technique of root loci is extended to mvcs instead of Nyquist criterion. Hence many of the comments on the previous methods are valid for this one also.

In spite of the above mentioned limitations of these methods they have been successfully used for some numerical and even practical problems as presented in the references quoted.

12.8 Scope of the Field

Even after giving highlights of some well known methods some major developments in the field of multivariable control theory remain untouched in this book. One of them is Wonham's geometrical approach(W-11). The other is the groups of techniques using no canonical form for the system description matrices. All these developments along with those described in the first eleven chapters of this book can be looked upon as parallel approaches having some advantages and some drawbacks. However it is believed that this book can provide one continuous stream of techniques sufficient to solve basic and more common problems of mvcs like pole assignment, decoupling, model matching, minimal realisation etc. The more sophisticated design specifications like insensitivity to parameter variation, disturbance rejection, integrity against failure are not discussed. But the degrees of freedom available in the design techniques explained in this book could be used to achieve these additional specifications.

Similarly system zeros are not discussed directly. However they are defined in Section 2.6. The zeros in the right half s-plane create difficulties in control(S-4,D-4). It can be proved that they are invariant under static precompensation and l.s.v.f.(M-4). Other than these two types of compensations can be used to locate the zeros arbitrarily. Cancellation of unwanted zeros can directly be implemented using any pole allocation method described in the book.

Model matching technique can directly help in arbitrary allocation of zeros. A good review of literature regarding the system zeros is given in the book edited by Fallside[F-3].

One more topic not discussed in detail is the construction of inverse systems[O-5,S-2,3,5,6,D-1]. The inverse system is not used as a part of compensator directly in any of the techniques. But the concept of right and left inverses is used as analytical tool in certain techniques.

12.9 Specific Approach and Contributions

Out of the many possible approaches to the study of multivariable control theory, the one accepted in this book is based mainly on the books of Wolovich[W-3], Fossard[F-1] and Layton[L-2] and partly on many other books and references listed at the end.

Many modifications are however introduced in the existing methods in order to simplify the calculations and notations, clarify the concepts, reveal the possible degrees of design freedom, reduce the number of restrictions and assumptions etc. The major aim was to present a logical development and systematic approach making it easy for understanding.

The main contributions offered by this book, first time, are :

i) The concept of using the dynamic feedback compensation of Chapter 10 as an independent frequency domain technique, rather than as a technique equivalent to l.s.v.f.

ii) The result, that this frequency domain technique can be used to diagonalise any proper transfer matrix.

iii) The method of solving model matching problem as presented here also has not been presented by anybody else before, as per the limited knowledge of the author. The model matching method given here is not in the form of a generalised technique, but is expected to be useful for some model matching problems. The method automatically separates the forward dynamic compensation, only when it is needed.

APPENDIX

(a) Properties of a Field $\mathcal{F}$:

A field is a set of elements (a, b, c ...) satisfying the following properties :

i) Commutative Property : $a + b = b + a$ is unique and belongs to $\mathcal{F}$.

ii) Associative property : $(a + b) + c = a + (b + c)$ belongs to $\mathcal{F}$.

iii) $a.(b.c) = (a.b).c$ belongs to $\mathcal{F}$.

iv) Elements 0 and 1 belong to $\mathcal{F}$, giving

$a + 0 = 0 + a = a$,

$a.1 = 1.a = a$.

v) Elements $(-a, -b, -c, \ldots)$ belong to $\mathcal{F}$.

vi) $a^{-1} = \frac{1}{a}$ belongs to $\mathcal{F}$, if $a \neq 0$, giving $a^{-1}\, a = a\, a^{-1} = 1$.

(b) Properties of commutative ring with identity :

It has all the properties of a field mentioned above except (vi).

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